should properly fall within the field of this association, and that discussions which affect such topics as projective geometry, second courses in calculus, the elementary theory of functions, and other courses commonly given to undergraduates, are properly subjects for discussion.

One more idea would seem to me to clarify the situation very materially. There have appeared in the MONTHLY from time to time articles which cannot be said to be of research character from the standpoint of the common acceptation of that word, but which nevertheless represent a great deal of labor of a purely investigational sort which would seem quite worthy of being called research in a broader interpretation of that word. This again is well illustrated by the historical papers mentioned above, all of which certainly constitute a very dignified form of research in this broader sense, though they may not satisfy the stricter interpretation placed ordinarily on the word research. The same thing can be said of a number of other papers which have appeared in the MONTHLY which deal essentially with college subjects. It is held that a dignified discussion which involves investigation from a scientific point of view is worthy of the name research in its broader application. While the new association will recognize fully the prior right of the American Mathematical Society in all questions which would ordinarily be termed research under the common interpretation, the attitude of this Association will be to encourage and to dignify all investigations of the character which have here been called research in the broader sense.

If I have tried to say what seems to me to be the policy of those responsible for the organization of the ASSOCIATION, I should perhaps add a word concerning the questions distinctly avoided thus far, which may be said to be not within our present intentions. One such which certainly deserves mention is the general notion of pedagogy in its more restricted interpretation. All of those questions which are termed pedagogical in the strict sense of that word have been held, and are held, by those responsible for the organization of the Association to belong to the field of education and to be wholly outside the field of the present association. Just as research will be held to be within its province only if the word is given a broad interpretation, it may be said also that the discussions which this association will foster may be termed pedagogical only if that word is used in a much broader sense than is common. I may define this broader sense to include those questions affecting instruction in which a *professional knowledge of the subject-matter* is a necessary element toward the formation of any dignified conclusion. That there is no doubt about the existence of such questions is amply proved by the files of the MONTHLY during the last two years.

This statement does not pretend to be exhaustive or infallible. The intention is to give as clear an idea as may be in a short space of characteristic topics which this Association will discuss. That the policy of the Association may be changed in the future and that the statements of this article are by no means binding upon the Association will be quite evident upon even a casual examination of the Constitution. E. R. HEDRICK, PRESIDENT.

COLUMBIA, MISSOURI, February 25, 1916.

"QUANTITY OF MATTER" IN DYNAMICS.[1]

By L. M. HOSKINS, Stanford University.

I. INTRODUCTORY.

1. *Different Views Regarding Mass.*—In explaining the significance of the quantity commonly called *mass* the term "quantity of matter" has been freely used by many writers on the laws of motion, including such high authorities as Newton, Maxwell, Kelvin, Tait and Clifford. These writers have assumed, either explicitly or tacitly, that the words have a meaning apart from dynamical laws. From this point of view the second Newtonian law (or "law of acceleration") is a statement of the way in which the *acceleration* of a body depends upon the two factors, (1) the *force* acting on the body and (2) the *quantity of matter* in the body. There are, however, those who dissent from this view, maintaining that the word mass as used in dynamics is fully defined by the law of acceleration itself; that the statement that "the mass of a body is a measure of its quantity of matter" contributes nothing to the definition. Because of these differing views it seems profitable to consider somewhat carefully what is really involved in the conception of quantity of matter.

In this study we may ask first what meaning, if any, may properly be attached to the words quantity of matter independently of the laws of dynamics, and secondly, whether the use of these words contributes anything to the definition of the quantity usually called mass in the formulation of dynamical principles.

2. *Postulates Regarding Acceleration and Force.*—To answer the second question requires an analysis of the logical import of the laws of motion; and such an analysis, if exhaustive, must involve not only the meaning of mass and force but also the question of bases of reference for estimating acceleration. The present object does not, however, require an exhaustive analysis of these questions. As regards acceleration, we shall assume as granted either the Newtonian view of absolute motion or the view that, at any rate, there exists a base for which the Newtonian laws are true.[2] As regards force, we shall treat separately two assumptions—(1) that force may be accepted as an exactly measurable magnitude independently of the laws of motion, and (2) that the definition of force as a measurable magnitude is contained in those laws; in accordance with these two assumptions parallel sets of propositions will be given, designed to illustrate the significance of "quantity of matter" in the law of acceleration.

II. PRELIMINARY QUANTITATIVE NOTIONS REGARDING MATTER.

3. *Definition of Body.*—We regard matter as consisting of individual parts which are indestructible; and we define a body as *any definite connected portion of matter.*

[1] Paper presented to the American Mathematical Society, August 3, 1915, at Stanford University, Calif.

[2] Such a base may be called a Newtonian base. (See W. H. Macaulay, *Mathematical Gazette*, October, 1900; *Bull. Am. Math. Soc.*, July, 1897, and April, 1898.)

Explanations of the laws of dynamics are often stated as applying to "particles" of matter or "material points." A more practical presentation must, however, deal with aggregates of matter of finite size. We shall here mean by a body any connected aggregate of matter; and when reference is made to the acceleration of a body, it is to be understood that all parts of the body have equal accelerations, *i. e.*, that the motion is a translation.

4. *Common Notion About Quantity of Matter.*—That a quantitative view of matter is one of the common notions associated with every-day experience is evident on a very little reflection. The statement that one of two gold coins contains twice as much of that particular material as the other is certainly not meaningless, quite apart from any consideration of inertia, or of weight, or of volume, or of value; although a precise explanation of the statement would be as difficult as would be a precise explanation of the statement that one body has twice the bulk of another. The notion of quantity, associated with particular kinds of matter, is in fact employed daily by the many persons who are engaged in buying and selling useful commodities. Whatever method may be used for the actual comparison of quantities, the object sought is to determine definite amounts of certain kinds of matter, the utility of which is not generally due to their inertia, or weight, or bulk.

5. *Definite Quantitative Notions Applicable to Matter.*—There are, moreover, certain very definite quantitative comparisons which are applicable to matter, without restriction to bodies of the same homogeneous substance, and without appeal to dynamical principles or to particular physical laws such as the law of gravitation. We recognize the applicability to matter of the notions that

(I) *The whole is greater than any part, and the whole is equal to the sum of its parts.*

And thus, quite independently of any kinetic considerations, we accept the truth of the following propositions:

(II) *If a body A be divided into two bodies B and C, the quantity of matter of A is greater than that of B or that of C; the sum of the quantities of matter of B and C is equal to that of A.*

(III) *If any two distinct bodies B and C be combined into a single body A, the quantity of matter of A is greater than that of B or that of C; the sum of the quantities of matter of B and C is equal to that of A.*

III. Significance of Quantity of Matter in the Law of Acceleration, Assuming Force to be Defined Non-kinetically.

6. *First Assumption Regarding Force.*—We now assume that force is a quantity which can be exactly measured, or at least defined as an exactly measurable magnitude, without any appeal to the laws of motion. Thus we may accept as satisfactory a method of comparing force-magnitudes based upon the deformations of springs or of any elastic bodies. It will be seen, however, that in the first set of illustrative propositions given below the only requirement regarding the measurement of forces is that it is possible to apply equal forces to any different

bodies; while in the second set it is assumed possible to apply forces of any desired relative magnitudes to any bodies at pleasure.

7. *The Effects of Equal Forces Applied to Different Bodies.*—To illustrate the way in which the notion of quantity of matter enters into our interpretation of the law of acceleration, we may state a number of propositions referring to suppositious cases in which equal forces act upon different bodies.

(*a*) If a force be applied to a body B, and an equal force be afterward applied to a body A formed by adding matter to B, the acceleration of A will be less than that of B.

(*b*) If, when equal forces are applied to two distinct bodies A and B, the acceleration of A is less than that of B, then by the removal of matter from A there may be produced a body A' such that equal forces applied to A' and B will cause equal accelerations; and by the addition of matter to B there may be produced a body B' such that equal forces applied to A and B' will cause equal accelerations.

(*c*) If two distinct bodies have equal accelerations when acted upon by equal forces, then an acceleration half as great will be caused by an equal force acting upon the body formed by combining the two.

(*d*) If any number of distinct bodies have equal accelerations when acted upon by equal forces, then an acceleration one nth as great will be caused by an equal force acting upon a body formed by combining n of the given bodies.

(*e*) If two distinct bodies, when acted upon by equal forces, have accelerations a' and a'', then the body formed by combining them will, if acted upon by an equal force, have an acceleration a such that $1/a = 1/a' + 1/a''$.

(*f*) If any number of distinct bodies, when acted upon by equal forces, have accelerations a', a'', $\cdots$, then the body formed by combining them will, if acted upon by an equal force, have an acceleration a such that $1/a = 1/a' + 1/a'' + \cdots$. (It will be noticed that this case really includes the five preceding.)

These propositions are consequences of the interpretation we put upon the following principle, which is a part of the law of acceleration:

(*A*) *Different bodies acted upon by equal forces have accelerations inversely proportional to their masses.*

A part of the import of proposition (*A*) undoubtedly consists in giving precision to the definition of mass; but if the meaning of mass were wholly contained in it, not one of the propositions (*a*), (*b*), (*c*), (*d*), (*e*) and (*f*) would necessarily follow from it. The reason we accept these as consequences of (*A*) is that we have, independently of (*A*), a conception of mass as a measure of quantity of matter and as satisfying propositions (I), (II) and (III).

8. *Case of Different Bodies Having Equal Accelerations.*—The part played by the quantitative notion of matter in our interpretation of the second law of motion is evident also in cases in which different bodies are supposed to be acted upon by forces such as to cause equal accelerations. This is brought out by the following propositions, which will be seen to have a correspondence to the propositions (*a*) $\cdots$ (*f*) given above.

(a') If a body A be formed by adding matter to a body B, the force required to give A any certain acceleration is greater than that previously required to give B an equal acceleration.

(b') If A and B are two distinct bodies such that a greater force must be applied to A than to B to cause equal accelerations, then by the removal of matter from A there may be produced a body A' such that equal forces must be applied to A' and B to cause equal accelerations; and by the addition of matter to B there may be produced a body B' such that equal forces must be applied to A and B' to cause equal accelerations.

(c') If two distinct bodies have equal accelerations when acted upon by equal forces, then a body formed by combining the two must be acted upon by twice as great a force in order to have an equal acceleration.

(d') If any number of distinct bodies have equal accelerations when acted upon by equal forces, then the body formed by combining any n of them must be acted upon by a force n times as great in order to have an equal acceleration.

(e') If the forces required to give two distinct bodies equal accelerations are F' and F'', then the body formed by combining the two must be acted upon by a force $F' + F''$ in order to have an equal acceleration.

(f') If the forces required to give any number of distinct bodies equal accelerations are F', F'', $\cdots$, then the body formed by combining them must be acted upon by a force $F' + F'' + \cdots$ in order to have an equal acceleration. (This includes the five preceding cases.)

These propositions are accepted as consequences of the following principle, which expresses part of the import of the law of acceleration:

(B) *In order that different bodies may have equal accelerations, they must be acted upon by forces whose magnitudes are proportional directly to the masses of the bodies.*

Assuming that force as a measurable magnitude is defined independently of this principle, a part of its import consists in giving precision to the meaning of mass; but if the whole meaning of mass were included in (B) no one of propositions (a') $\cdots$ (f') would necessarily follow from it. We accept these as consequences of (B) because we have, independently of (B), the conception that mass is a measure of quantity of matter and satisfies propositions (I), (II) and (III).

9. *Effects of Different Forces Applied at Different Times to the Same Body.*—Besides principles (A) and (B), each of which expresses a part of the import of the law of acceleration, there is a third principle of equal interest which may be stated as follows:

(C) *Different forces applied at different times to the same body cause accelerations having the same ratios as the magnitudes of the forces.*

This is not independent of (A) and (B); in fact, careful reflection shows that any one of the three propositions (A), (B) and (C) may be inferred from the other two.

10. *General Statement of Law of Acceleration.*—The general principle of which (A), (B) and (C) are particular cases may be stated as follows:

(*D*) *Any bodies acted upon by any forces will have accelerations proportional directly to the force-magnitudes and inversely to the masses* (*quantities of matter*) *of the bodies.*

Principle (*D*) may be called the general law of acceleration; it is equivalent to Newton's second law of motion, so far as this refers to forces acting singly.

IV. Significance of Quantity of Matter When Force is Defined Kinetically.

11. *Second View of Force.*—It is generally recognized that, while the measurement of forces by their effects in deforming elastic bodies (as by the spring balance) is of very great practical use, the exact comparison of force-magnitudes for the purposes of dynamics must make use of the law of acceleration. If this view is adopted a different form must be given to the explanation of the rôle of mass or quantity of matter in the laws of motion. To this we proceed.

12. *Formulation of the Laws of Motion.*—A rigorous logical analysis shows that the second and third laws of Newton (the law of acceleration and the law of action and reaction) are not independent principles. It will be seen, however, that the import of both is covered by the following propositions:

(1) *The acceleration of a body is always due to the influence of other bodies.*

(2) *If the acceleration of a body is due to the influence of more than one other body, it is equal to the vector sum of components, each due to some one of the bodies.*

(3) *If A and B are any two individual portions of matter, the acceleration of A due to B and that of B due to A are always oppositely directed, and their magnitudes are in a constant ratio.*

Definition.—The term "acceleration-ratio of A with respect to B" will be used to denote the ratio:

$$(\text{acceleration of } A \text{ due to } B)/(\text{acceleration of } B \text{ due to } A).$$

(4) *If A, B and C are any three distinct bodies, the acceleration-ratio of A with respect to B is equal to the acceleration-ratio of A with respect to C divided by the acceleration-ratio of B with respect to C.*[1]

[1] In connection with (3) it should be added that if A and B are "particles,"—i. e., if their dimensions are vanishingly small in comparison with their distance apart,—their mutually-caused accelerations are directed along the line AB.

The "law of composition," expressed by (2), is sometimes held to mean that the acceleration of a body A due to a body B is not influenced by any third body C; in other words, that if a' is the acceleration of A due to B when C is absent, and a'' the acceleration of A due to C when B is absent, then when both B and C are present the acceleration of A will be the vector sum of a' and a''. This cannot be accepted as a valid interpretation of the law of composition; all that is implied by this law is that the acceleration of every particle can at every instant be resolved into components regarded as individually "due to" other particles in the sense that they satisfy (2), (3) and (4). The supposition that the acceleration of A due to B may be changed by the presence of C is entirely consistent with these laws. (See K. Pearson, "The Grammar of Science," second edition, page 317.)

The import of the laws is, in fact, summed up in the statement that the total momentum of a system of bodies is not changed by the action of these bodies upon one another (total momentum being understood in its broadest sense as a localized vector quantity). The definition of momentum must of course depend upon that of mass given in 13.

13. *Definition of Mass.*—These principles imply that *it is possible to assign to all bodies individual constant numbers such that the acceleration-ratio of any two bodies is the inverse ratio of their assigned constants.* Constants satisfying this condition are called the *masses* of the bodies. The actual values of the mass-constants are fixed as soon as any one of them is assigned; this is equivalent to choosing a unit of mass.[1]

14. *Mass as Quantity of Matter.*—That there is any relation between the quantity thus defined as mass and matter regarded as a quantity is not immediately apparent from principles (1), (2), (3) and (4). That there is such a relation, and that mass as just defined is an appropriate measure of quantity of matter, is illustrated by the following propositions referring to suppositious cases. It will be seen that there is a correspondence between these propositions and those above designated as $(a) \cdots (f)$.

(a'') If a body A be formed by adding matter to a body B, then the acceleration-ratio of A with respect to any third body D will be less than that of B with respect to D.

(b'') If A and B are two distinct bodies such that the acceleration-ratio of A with respect to a third body is less than that of B with respect to that body, then by the removal of matter from A there may be produced a body A' such that the acceleration-ratios of A' and B with respect to any third body are equal; and by the addition of matter to B there may be produced a body B' such that the acceleration-ratios of A and B' with respect to any third body are equal.

(c'') If the acceleration-ratios of two bodies B and C with respect to a third body D have equal values r, then if B and C be combined into a single body A the acceleration-ratio of A with respect to D will be $r/2$.

(d'') If there be any number of bodies having equal acceleration-ratios r with respect to a body D, then if any n of them be combined into a single body its acceleration-ratio with respect to D will be r/n.

(e'') If two distinct bodies have acceleration-ratios r', r'' with respect to another body D, then the body formed by combining the two will have with respect to D an acceleration-ratio r such that $1/r = 1/r' + 1/r''$.

(f'') If any number of distinct bodies have, with respect to a body D, the acceleration-ratios r', r'', $\cdots$, then the body formed by combining them will have, with respect to D, an acceleration-ratio r such that $1/r = 1/r' + 1/r'' + \cdots$. (This includes the five preceding cases.)

These propositions are consequences of the principle that

(E) *The acceleration-ratios of any bodies A, B, C, $\cdots$, with respect to any one other body are in the inverse ratios of the quantities of matter of A, B, C, $\cdots$.*

A part of the import of (E) is a precise *definition* of quantity of matter; but unless we had, independently of (E), a quantitative conception of matter satisfy-

[1] The above method of formulating the laws of motion and of defining mass is substantially that which has been used by a number of writers, among whom may be mentioned E. MACH, K. PEARSON, G. KIRCHHOFF, P. APPELL, A. E. H. LOVE, W. H. MACAULAY. Some of these authors have explicitly recognized that the conception of mass which we actually use includes the notion of a quantitative measure of matter satisfying propositions (I), (II) and (III).

ing propositions (I), (II) and (III), we should not recognize that (a''), $\cdots$ (f'') are consequences of (E); and we certainly should not recognize that they are consequences of (1), (2), (3) and (4).

Proposition (E) shows that the quantity above defined as *mass* is an appropriate measure of *quantity of matter*.

15. *Definition of Force.*—From the present point of view the import of the Newtonian laws is completely covered by principles (1), (2), (3), (4) and (E); and it will be noticed that the word force is not used in the statement of these propositions. A definition of force may, however, be associated with them. Thus (1) and (2) suggest the following:

A force is an action exerted by one body upon another, the effect of which is to give the latter body an acceleration-component.

With this definition it is seen that (2) expresses the law of vector composition of forces.

An exact quantitative definition of force, consistent with the above principles, may be stated as follows:

A force is an action exerted by one body upon another, measured by the product of the acceleration produced and the mass of the body acted upon.

With this definition, and that of mass given above, principle (3) is seen to be equivalent to the law of action and reaction in its ordinary form:

Whenever one body exerts a force upon another, the second exerts an equal and opposite force upon the first.

V. Mass in the Law of Gravitation.

16. *Laws of Dynamics Independent of Law of Gravitation.*—Because the weights of bodies in the same locality are proportional to their masses, weight[1] plays an important part not only in the practical work of comparing the masses of bodies but in explanations of the principles of dynamics. It must not be forgotten, however, that the laws of dynamics are quite independent of the law of gravitation.

17. *Significance of Mass in the Law of Gravitation.*—The law of gravitation states that every portion of matter exerts an attractive force upon every other portion, and that the magnitude of the attractive force exerted by one particle[2] upon another depends in a definite way upon the masses of the particles and their distance apart. It is of interest to remark that all the questions which have been raised regarding the significance of mass in the law of acceleration might be raised with equal pertinence with regard to the meaning of mass in the law of gravitation. Thus the statement that "two particles which are at equal distances

[1] The word weight is often used popularly in the sense of quantity of matter, and some writers recommend the general adoption of this usage. There would be no objection to this if it did not lead to the confusing of two distinct things; but it is necessary to use some word to denote the gravitational pull of the earth upon a body, and scientific usage is nearly unanimous in assigning this meaning to the word weight. This usage is followed in the present paper.

[2] It is convenient here to refer to particles rather than to bodies of finite dimensions in order to avoid complexities of statement due to the fact that forces combine by vector addition.

from a third particle are attracted by it with forces proportional to their masses" might be held to be merely a definition of mass. It might be asserted that "the statement that A's mass is three times as great as B's means merely that if A and B are equally distant from C the attractive force of C upon A is three times that of C upon B."

It is not difficult to show, however, that the conception of mass as a measure of quantity of matter is an essential part of the interpretation we put upon the law of gravitation. Thus if, when A and B are equally distant from C, the attraction of C upon A is three times that of C upon B, we say that it is because A has three times as much matter as B; and we infer that A might be divided into three parts each having a mass equal to that of B.

The law of gravitation thus derives an essential part of its significance from a notion of mass which is independent of that law—the notion that mass is a measure of the matter of which bodies are composed and satisfies the fundamental quantitative relations expressed by (I), (II) and (III). At the same time it must be recognized that the law itself gives precision to the definition and furnishes a method of comparing quantities of matter with exactness.

18. *Agreement of Two Methods of Estimating Quantity of Matter.*—It has been seen that the law of acceleration and the law of gravitation furnish two independent methods of applying exact quantitative measurements to matter. That the results of the two methods are in apparently exact agreement is a matter of experimental knowledge rather than of *a priori* necessity. If the fact were otherwise, weighing would not be an available method of comparing the masses of bodies for the purposes of dynamics.

GEOMETRICAL AND OTHER ILLUSTRATIONS OF INDETERMINATE FORMS.

By W. V. LOVITT, Purdue University.

In the AMERICAN MATHEMATICAL MONTHLY, Vol. III, 1893, pp. 207–8, there is given by B. F. Finkel a geometrical proof that $0 \times \infty$ is indeterminate. In the present paper additional illustrations, geometrical and otherwise, are given of this and other indeterminate forms.

I. The area generated by a segment of a straight line AB (Fig. 1) revolving about an axis CD in its plane is given by the formula:

$$\text{Area } AB = 2\pi \cdot AB \cdot MO = 2\pi \cdot CD \cdot MR,$$

where CD is the projection of the segment AB upon the line CD, M is the middle point of AB, O is the center of the segment CD, and MR is perpendicular to AB. Let AB revolve about M and become perpendicular to CD. Then MR becomes parallel to CD and $CD \cdot MR$ becomes $0 \times \infty$.

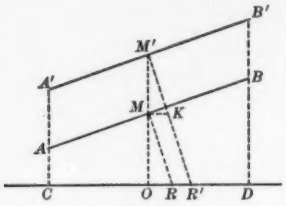


FIG. 1.

But $$CD \cdot MR = AB \cdot MO = K_1 \text{ (constant)}.$$

Let us take a line segment $A'B' = AB$ and parallel to AB. Let M' be the middle point of $A'B'$, $M'R'$ perpendicular to $A'B'$ and parallel to MR. Then

$$\text{Area } A'B' = 2\pi \cdot CD \cdot M'R'.$$

Rotate $A'B'$ about M' until it is perpendicular to CD. Then $CD \cdot M'R'$ becomes $0 \times \infty$. But

$$CD \cdot M'R' = A'B' \cdot M'O = K_2 \text{ (constant)}.$$

We note that $K_1 \neq K_2$; that is, our form $0 \times \infty$ takes on different values in these two instances and can thus be made to vary at will.

Draw MK parallel to CD. Consider the difference

$$\text{Area } A'B' - \text{Area } AB = 2\pi CD \cdot (M'R' - MR) = 2\pi CD \cdot M'K.$$

Let $A'B'$ and AB become perpendicular to CD. Then $M'R'$ and MR become parallel to CD and

$$M'R' - MR \text{ becomes } \infty - \infty.$$

But

$$M'R' - MR = M'K$$

and $M'K$ becomes infinite as $A'B'$ becomes perpendicular to CD. Hence, in this case the indeterminate form $\infty - \infty$ becomes infinite.

Incidentally, we have in the product $CD \cdot M'K$ an indeterminate form of the type $0 \times \infty$.

An additional illustration of the form $\infty - \infty$ is furnished by the hyperbola. This curve is defined as the locus of a point P, the difference of whose distances from two fixed points, F' and F, is a constant, $2a$. That is

$$F'P - FP = 2a.$$

As $F'P$ increases indefinitely, FP also increases indefinitely and

$$F'P - FP \text{ takes the form } \infty - \infty.$$

II. Consider now (Fig. 2) a triangle with a fixed base a, with sides x and y as indicated, the vertex P being on a line l which is parallel to the base. The

area of this triangle is constant and will be denoted by c. Then

$$c = \frac{a}{2} x \sin\theta, \quad \text{and} \quad \frac{2c}{a} = x \sin\theta = x/\csc\theta.$$

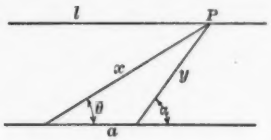


FIG. 2.

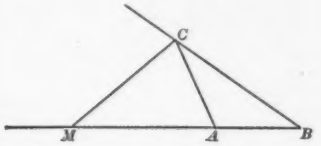


FIG. 3.

As P runs off to infinity on the line l, $x \sin\theta$ becomes $\infty \times 0$, while $x/\csc\theta$ becomes ∞/∞. From the law of sines,

$$\frac{x}{y} = \frac{\sin\alpha}{\sin\theta}.$$

As P moves off to infinity on the line l, the left and right hand members of the last equation take on the forms ∞/∞ and $0/0$ respectively. Thus we have two illustrations of the change from one indeterminate form to another.

III. The following theorem from plane geometry furnishes examples of the indeterminate forms ∞/∞ and $\infty - \infty$.

The bisector CM (Fig. 3) of an exterior angle of a triangle ABC divides the opposite side externally into segments which are proportional to the adjacent sides. That is,

$$\frac{MA}{MB} = \frac{CA}{CB}.$$

Let BC and AB remain fixed in length, and let AC increase, approaching BC in length. Then CM becomes parallel to AB, while MA/MB becomes ∞/∞. But

$$\lim_{CA \to BC} \frac{MA}{MB} = \lim_{CA \to BC} \frac{CA}{BC} = 1.$$

The difference $MB - MA$, which is always equal to the constant AB, becomes $\infty - \infty$.

IV. An additional illustration of the form $0 \times \infty$ is furnished by inversion with respect to a given circle, $x^2 + y^2 = a^2$. Let P and P' be inverse points and O the center of the given circle, then

$$\overline{OP} \cdot \overline{OP'} = a^2.$$

As OP approaches zero, OP' becomes indefinitely large and $\overline{OP} \cdot \overline{OP'}$ becomes $0 \times \infty$. In particular this form occurs in finding the inverse of a circle through the origin. The inverse is a straight line.

V. We add here two problems taken from works on hydraulics.

The theoretical efflux Q (Fig. 4) through a triangular orifice in a thin vertical

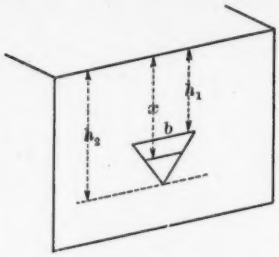


FIG. 4.

plate or wall, with base, b, horizontal, is given by the following formula:

$$Q = \frac{2b\sqrt{2g}}{15} \frac{[2h_2^{5/2} - 5h_2h_1^{3/2} + 3h_1^{5/2}]}{(h_2 - h_1)}.$$

Let $h_1 = h_2$; then certainly $Q = 0$, while the expression for Q in the formula takes on the indeterminate form 0/0.

Again, in hydrostatics the head h' on the center of pressure of a submerged rectangular plane, with edges parallel to the surface, one end having a head h_2, the other a head h_1, is

$$h' = \frac{2}{3} \cdot \frac{h_2^3 - h_1^3}{h_2^2 - h_1^2} \cdot = \frac{2}{3}\left(\frac{h_2 - h_1}{h_2 - h_1}\right)\left(\frac{h_2^2 + h_2h_1 + h_1^2}{h_2 + h_1}\right)$$

When $h_2 = h_1$, $h' = h$, while the expression for h' in the formula becomes 0/0.

VI. We conclude with a few simple arithmetical illustrations.

(1) Type $\frac{0}{0}$: $\frac{1}{9}; \frac{.1}{.9}; \frac{.01}{.09}; \cdots$. Constantly equal to $\frac{1}{9}$.

(2) Type $\frac{\infty}{\infty}$: $\frac{3}{10}; \frac{33}{100}; \frac{333}{1000}; \cdots$. Approaches $\frac{1}{3}$.

Any repeating decimal furnishes an additional example.

(3) Type $\infty - \infty$: $1.3 - 1; 2.33 - 2; 3.333 - 3; \cdots$. Approaches $\frac{1}{3}$.

(4) Type $0 \times \infty$: $1.2; \frac{1}{2} \cdot 4; \frac{1}{4} \cdot 8; \cdots$. Constantly equal to 2.

(5) Type ∞^0: $(2^2)^{1/2}; (2^3)^{1/3}; (2^4)^{1/4}; \cdots$. Constantly equal to 2.

(6) Type 0^0: $(2^{-2})^{1/2}; (2^{-3})^{1/3}; (2^{-4})^{1/4}; \cdots$. Constantly equal to $\frac{1}{2}$.

A TRIBUTE TO SAMUEL WALKER SHATTUCK.

Professor Shattuck was administrative head of the department of mathematics in the University of Illinois for nearly forty years (1868–1906), having come to the University as the first and the only instructor in mathematics when the entire faculty of the institution consisted of but eleven members. He lived to see the time when the faculty of his department was twice as large as the entire faculty of the University had been when he entered it. The remarkable growth of the University during the years that he was actively connected with it was largely due to his devotion and integrity, for he was not only in charge of the mathematical department during the greater part of this period but also very influential in directing the finances of the institution, acting as business agent and manager during the years 1873–1905, and as comptroller from 1905 to 1912 when he retired on a Carnegie pension on account of failing health.

Notwithstanding the fact that his university duties were unusually heavy and that he performed unselfishly the many other services to society which his prominence in the community made possible, he found time to write a mimeographed work on calculus, consisting of two volumes and giving evidences of much thought. He was regarded as a good teacher both in the classroom and outside of it, for his training as a soldier gave him a dignified bearing which led the former President Draper to remark that "if he did nothing else it would be worth while to the state of Illinois to pay him his salary just to have him on the campus as a visible example to young men."

The great and faithful services which Professor Shattuck rendered the University of Illinois during a long period of years helped to endear him to all interested in its welfare, but his pleasing personality combined with a readiness to be helpful in every good work and to stand for justice and the strictest impartiality added very much to this endearment.

During the last few years of his life he witnessed numerous evidences of appreciation on the part of the alumni and the faculty, and since his death on February 13, 1915, two appreciative biographical sketches, relating to his services, written by Professor S. A. Forbes and Dean T. A. Clark respectively, have appeared in the *Alumni Quarterly* of the University of Illinois.

Professor Shattuck was an army officer during the Civil War, and belonged to a family of soldiers, his father, grandfather, and great-grandfather having all been officers in the colonial or national armies. He was born at Groton, Massachusetts, on February 18, 1841, and came to the University of Illinois from Norwich University, Vermont, where he had been professor of mathematics and military tactics. Although he entered the faculty of the University of Illinois less than half a century ago it is interesting to note that he was appointed assistant professor of mathematics, instructor in military tactics, and first commander of the University corps of cadets. These appointments indicate that the professor in the University of Illinois in those days did not enjoy sufficient leisure to become the scholar which the more recent opportunities have made

possible. The varied duties of these early professors tended, however, to bring the students and the faculty into more frequent contact and thus they enabled the unusually strong members of the faculty to leave a more lasting impression on the minds and the hearts of their students than is possible under the more modern conditions.

In his method of teaching, Professor Shattuck exhibited his usual willingness to be helpful to all who were likely to profit by such help, but he expected the student to do his part first. That is, he was willing to help all those who had made reasonable efforts to follow the explanations of the textbook and who had met with difficulties, but he did not, as a rule, lecture to his classes on the general bearing of the subject under consideration. His examination questions were usually rather difficult but he was not very severe in grading the answers. His students knew that they were dealing with a man who expected them to do serious work and who was likely to make few concessions for irregularities.

While his many other duties prevented Professor Shattuck from taking an active part in the development of mathematics, he frequently exhibited a keen interest in these developments. In particular, he was one of the first to join the American Mathematical Society and he continued this membership to the end of his life. As evidence of his deep interest in teaching we may cite the fact that when the President of the University suggested to him that he might devote himself entirely to the financial interests of the University the President received the reply: "When I give up teaching I shall give up the University."

In brief, Professor Shattuck devoted himself without ostentation but with complete devotion to what appeared to him to be the most important work of the time. When, as a young man, his country seemed to need his services in the defense of a great principle he enlisted twice in the United States Army and bore bravely the hardships of the battlefield and the serious wound received while at the front. When he came to the University of Illinois and found that a few men had to shoulder a large number of important outside duties he assumed his full share and performed all his work punctually and with great care. As a member of the church, as a citizen, and as the head of a family he led an exemplary life, and thus extended his teaching beyond his classroom. In particular, his careful management of the finances of the University inspired widespread confidence and this confidence has been one of the greatest assets in winning for the institution the loyal support which has in recent years been given to it by the state.

G. A. MILLER.

UNIVERSITY OF ILLINOIS.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

Send all communications to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

ALGEBRA.

450. Proposed by J. E. ROWE, Pennsylvania State College.

If the four roots of the quartic equation $A \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, are so related that $B \equiv a_0a_4 - 4a_1a_3 + 3a_2^2 = 0$, show by elementary algebra that two roots of A are real and two imaginary. Show also by means of elementary algebra that A cannot have two equal roots without having three, if the condition $B = 0$ is satisfied.

451. Proposed by H. S. UHLER, Yale University.

Prove that

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cos\frac{x}{2^4}\cdots.$$

GEOMETRY.

481. Proposed by PAUL CAPRON, U. S. Naval Academy.

Show that the locus of the intersection of a pair of perpendicular normals to a parabola $y^2 = 4px$ is the parabola $y^2 = p(x - 3p)$.

482. Proposed by ROBERT G. THOMAS, The Citadel, Charleston, S. C.

In laying out a kite-shaped mile race-track, composed of a circular arc and two intersecting tangents at the ends of the arc, determine the angle at the center of the arc (*a*) when the length of the arc equals the sum of the two tangents, and (*b*) when the arc is equal to the length of each tangent.

CALCULUS.

402. Proposed by C. N. SCHMALL, New York City.

If (x, y) be a double point on the curve $u \equiv f(x, y) = 0$, show that (1) the two branches of the curve will cut orthogonally if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0;$$

and (2) if this point be made the origin, then the equation of the tangents to the branches will be

$$(y'^2 - x'^2)\frac{\partial^2 u}{\partial x^2} + 2x'y'\frac{\partial^2 u}{\partial x\,\partial y} = 0,$$

where (x', y') are the current coördinates of points on the tangents.

NOTE.—In an early issue, we will publish all the unsolved problems in Number Theory proposed from January, 1913, to December, 1915. EDITORS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

439. Proposed by A. M. KENYON, Purdue University.

If k, n are natural numbers, $n > 2k$, show that

$$\frac{2^k}{\lfloor k}\sum_{i=0}^{I\left(\frac{n+1}{2}\right)}\frac{1}{\lfloor 2i+1 \rfloor \lfloor n-k-2i}} = \frac{2^n}{\lfloor n+1}\sum_{i=0}^{k}\binom{n-i}{n-k},$$

where $I(n/2)$ denotes the integral part of $n/2$ and $\binom{n}{k}$ is the coefficient of x^k in $(1 + x)^n$.

SOLUTION BY FRANK IRWIN, University of California.

On the right side

$$\sum_{i=0}^{k}\binom{n-i}{n-k}=\binom{n+1}{n-k+1}.$$

This formula expresses in symbols the well-known property of the Pascal triangle, that any term is equal to the sum of all terms above it in the preceding column. (See, for instance, LUCAS, *Théorie des Nombres*, page 6.)

Or it may be proved as follows:

$$\sum_{i=0}^{k}\binom{n-i}{n-k}=\sum_{i=0}^{k}\left[\binom{n-i+1}{n-k+1}-\binom{n-i}{n-k+1}\right]=\sum_{i=0}^{k}\binom{n-i+1}{n-k+1}-\sum_{i=1}^{k+1}\binom{n-i+1}{n-k+1}.$$

Here all the terms but one cancel in pairs, leaving $\binom{n+1}{n-k+1}$.

On the other hand, the left side of our given equation may be written

$$\frac{2^k}{\lfloor n-k+1\rfloor\lfloor k}\sum_{i=0}^{I\left(\frac{n+1}{2}\right)}\binom{n-k+1}{2i+1}.$$

Here the summation sign takes in the 2d, 4th, $\cdots$, *all* the even-placed coefficients,

$$\binom{n-k+1}{1},\quad\binom{n-k+1}{3},\ \cdots$$

in the development of $(1+x)^{n-k+1}$; the sum of which is well known to be 2^{n-k}.

Our formula, then, reduces to the obviously true form:

$$\frac{1}{\lfloor n-k+1}\,\frac{2^k}{\lfloor k}\,2^{n-k}=\frac{2^n}{\lfloor n+1}\binom{n+1}{n-k+1},$$

Also solved by the Proposer.

440. Proposed by W. D. CAIRNS, Oberlin College.

n being a positive integer, find the sum of the series

$$2n^2+4(n-1)^2+2(n-2)^2+4(n-3)^2+2(n-4)^2+\cdots,$$

where the succeeding coefficients are alternately 4 and 2; or, more generally, the series

$$an^2+b(n-1)^2+a(n-2)^2+b(n-3)^2+a(n-4)^2+\cdots.$$

L'Intermédiaire, July, 1913.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

The problem as originally published in the September, 1915, issue contained two misprints. The question is indefinite. These series are not convergent, and do not break off after a finite number of terms. However, it is easy to find the sum of $2k$ terms.

Expanding, we can write the second series as

$$\begin{aligned}S&=a\{n^2+n^2-4n+4+n^2-8n+16+n^2-12n+36+\cdots+n^2-2(2k-2)n+(2k-2)^2\}\\&\quad+b\{n^2-2n+1+n^2-6n+9+n^2-10n+25+\cdots+n^2-2(2k-1)n+(2k-1)^2\}\\&=a\{kn^2-4n(1+2+3+\cdots+k-1)+(2^2+4^2+6^2+\cdots+[2k-2]^2)\}\\&\quad+b\{kn^2-2n(1+3+5+\cdots+2k-1)+(1^2+3^2+5^2+\cdots+[2k-1]^2)\}.\end{aligned}$$

Summing the series in parentheses we obtain

$$S = a\left\{kn^2 - 2nk(k-1) + \frac{4k^3 - 6k^2 + 2k}{3}\right\} + b\left\{kn^2 - 2nk^2 + \frac{4k^3 - k}{3}\right\},$$

which may be written as

$$(a+b)kn^2 - 2nk(ak + bk - a) + \left\{\frac{4}{3}k^3(a+b) - 2ak^2 + \frac{k}{3}(2a - b)\right\}.$$

Also solved by H. C. FEEMSTER, HORACE OLSON and the PROPOSER.

441. Proposed by W. D. CAIRNS, Oberlin College.

Prove that the equation $(e-1)x = e^x - 1$ has two and only two real roots.

I. SOLUTION BY H. S. UHLER, Yale University.

Let $y = e^x - 1 - (e-1)x$ and observe that $y = 0$ for $x = 0$ and $x = 1$. It remains to show that there can be no more real roots.

$$\frac{dy}{dx} = e^x - e + 1, \quad (1) \qquad \frac{d^2y}{dx^2} = e^x. \quad (2)$$

Equation (1) shows that the slope of the tangent is positive for all values of x greater than $\log_e(e-1)$ and negative for all values of x less than this value ($x_0 = 0.541325$). Since e^x is essentially positive, equation (2) indicates formally that the single stationary point, indicated by equation (1), corresponds to a minimum value of y. The coördinates of the minimum are $x_0 = \log_e(e-1)$ and $y_0 = (e-1)(1-x_0) - 1 = -0.211867$. It is clear, therefore, from the properties of the graph that the curve cannot cut the axis of x in more than two points and hence the given equation has two and only two real roots.

II. SOLUTION BY GRACE M. BAREIS, Ohio State University.

Writing e in series form in each member and transposing, the equation becomes

$$x\left[\frac{x-1}{\lfloor 2} + \frac{x^2-1}{\lfloor 3} + \frac{x^3-1}{\lfloor 4} + \cdots\right] = 0, \text{ or } x(x-1)\left[\frac{1}{\lfloor 2} + \frac{x+1}{\lfloor 3} + \frac{x^2+x+1}{\lfloor 4} + \cdots\right] = 0.$$

Hence, $x = 0$, $x - 1 = 0$, or $\frac{1}{\lfloor 2} + \frac{x+1}{\lfloor 3} + \frac{x^2+x+1}{\lfloor 4} + \cdots = 0$. Since each term of the left member of the last equation is positive, this equation can have no real positive root.

To prove that the original equation can have no real negative root, put it in the form

$$x = \frac{e^x - 1}{e - 1},$$

or $x = e^{x-1} + e^{x-2} + e^{x-3} + \cdots$, a convergent series. Any real negative number makes the left member negative but the right member positive, and hence there are no real negative roots. Hence, 0 and 1 are the only real roots.

Solutions were also received from W. L. AGARD, F. L. GRIFFIN, H. C. FEEMSTER, WALTER C. EELLS, HORACE OLSON, C. E. HORNE, IRBY C. NICHOLS, and FRANK IRWIN.

GEOMETRY.

467. Proposed by E. T. BELL, Seattle, Washington.

It is well-known that if i, j, k, l are concyclic points, W_i the Wallace line (frequently, and erroneously, called the Simson line), of i with respect to the triangle jkl, then W_i, W_j, W_k, W_l are concurrent, say in the point $\{i, j, k, l\}$. If $1, 2, 3, \cdots$ denote concyclic points, prove that:

(i) $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$, $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$ are concyclic; say on the circle $[1, 2, 3, 4, 5]$;

(ii) Starting with 1, 2, 3, 4, 5, 6, omitting each point in turn, by (i), six circles, are found; these are concurrent, say in the point $\{1, 2, 3, 4, 5, 6\}$;

(iii) Starting with 1, 2, 3, 4, 5, 6, 7, seven points of the kind in (ii) are found; these lie on a circle.

(iv) Continuing thus indefinitely, there is, in each case, finally a unique point or circle according as the number of initial points is even or odd. Also, at any stage, the point of concurrence on the circle bears a simple relation to the initial points: what is it?

I. Solution by J. W. Clawson, Collegeville, Pa.

If H_i is the orthocenter of triangle jkl, O the center of the circle, radius R, on which the points i, j, k, l lie,—

iH_i = twice distance of O from $kl = jH_i$. Also iH_j is parallel to jH_i; hence H_iH_j is equal to, and parallel to, ij. It follows that $H_iH_jH_kH_l$ is inversely congruent with $ijkl$, the corresponding sides of the quadrilaterals being equal and parallel, but arranged in opposite orders.

Again, iH_i, jH_j, kH_k, lH_l have a common mid-point; and, since iH_i bisects W_i and so on, this common point is the point of concurrency of W_i, W_j, W_k, W_l. The point of concurrency $\{i, j, k, l\}$ bisects the join of O to the center of the circle circumscribing $H_iH_jH_kH_l$.

(i) If H_{123} is the orthocenter of triangle 123, H_{123}, H_{234}, H_{341}, H_{412} are four vertices of a pentagon inversely congruent with 12345. Moreover the sides of this pentagon are parallel to the corresponding sides of the pentagon 12345. Call the center of the circumcircle of this pentagon, P_5. Similarly H_{123}, H_{235}, H_{351}, H_{512} and three other sets of orthocenters determine four other congruent pentagons with parallel sides. Each of the orthocenters occurs as a vertex of two of the pentagons. It follows that P_5, P_4, P_3, P_2, P_1, the centers of the circles circumscribing these five pentagons, form a pentagon which is congruent with each of the pentagons and hence inversely congruent with the original pentagon 12345. The points P_5, P_4, P_3, P_2, P_1 are therefore concyclic. Call the center of their circle, O_6.

It follows at once that the points (1234), (2345), (3451), (4512), (5123) are concyclic on a circle of half the radius of the given circle, its center bisecting the join of O to O_6.

(ii) The circle, center O_6, passes through P_5, the center of a circle determined by the orthocenters of the triangles obtained by selecting triads of points from 1, 2, 3, 4. In the same way if we consider the five points 1, 2, 3, 4, 6 we shall obtain a circle, center O_5, which passes through P_5 and through four new points. In this way we can get six hexagons, one of which has P_1, P_2, P_3, P_4, P_5 for five of its vertices, each inversely congruent with the hexagon 123456 and five vertices of each hexagon (orthocenters of triangles obtained by selecting three points from the six) belonging also to a different hexagon of the system. It follows that $O_6O_5O_4O_3O_2O_1$ forms a hexagon which is congruent with each of the hexagons and hence is inversely congruent with the original hexagon 123456. Hence $O_6, O_5, O_4, O_3, O_2, O_1$ are concyclic; and the circles with $O_6, O_5, O_4, O_3, O_2, O_1$ as centers and with R as radius concur at a point, say Q_7.

It follows at once that circles (1, 2, 3, 4, 5), $\cdots$ concur at a point halfway between Q_7 and O.

(iii) This process can be continued indefinitely as stated in the problem.

II. Solution by Norman Anning, Chilliwack, B. C.

Letting $e^{i\phi} = \cos\phi + i\sin\phi$, then $Re^{i\theta_j}$ $(j = 1, 2, 3, 4 \cdots)$ are points on a circle of radius R, whose center is at the origin. The W line of 4 quâ 123 passes through the foot of the perpendicular from 4 on line 12 and makes with this perpendicular an angle equal to that between 13 and 14. All points on W_4 are given by

$$Re^{i\theta_4} + ke^{i\frac{\theta_1+\theta_2}{2}} + xe^{i\frac{\theta_1+\theta_2+\theta_3-\theta_4}{2}},$$

where x is a running coördinate on the line and k, the distance from 4 to 12, is to be determined. Projecting on the radius vector midway between 1 and 2,

$$k + R\cos\left(\theta_4 - \frac{\theta_1+\theta_2}{2}\right) = R\cos\frac{\theta_1-\theta_2}{2}, \qquad k = 2R\sin\frac{\theta_4-\theta_2}{2}\sin\frac{\theta_4-\theta_1}{2}.$$

$$W_4 \text{ is } Re^{i\theta_4} + 2R\sin\frac{\theta_4-\theta_2}{2}\sin\frac{\theta_4-\theta_1}{2}e^{i\frac{\theta_1+\theta_2}{2}} + xe^{i\frac{\theta_1+\theta_2+\theta_3-\theta_4}{2}}.$$

$$W_3 \text{ is } Re^{i\theta_3} + 2R\sin\frac{\theta_3-\theta_2}{2}\sin\frac{\theta_3-\theta_1}{2}e^{i\frac{\theta_1+\theta_2}{2}} + ye^{i\frac{\theta_1+\theta_2-\theta_3+\theta_4}{2}}.$$

To find the common point of these lines equate the two expressions, separate reals and imaginaries and solve for x and y. Result:

$$x = y = 2R\cos\frac{\theta_4 + \theta_3 - \theta_2 - \theta_1}{2}.$$

When the exponential values are inserted for the sines and cosines, the expression for the common point of W_4 and W_3 becomes

$$\frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4}).$$

The symmetry of this expression shows that the point is the common point of the four W lines.

$$\{1, 2, 3, 4\} \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} - e^{i\theta_5}),$$

$$\{1, 2, 3, 5\} \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} - e^{i\theta_4}),$$

. . . . . .

$$\{2, 3, 4, 5\} \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} - e^{i\theta_1}).$$

These 5 points lie on the circle whose center is the point, $\frac{R}{2}\sum_{p=1}^{p=5} e^{i\theta_p}$, and whose radius is $R/2$. Where ϕ is variable, the circle

$$[1, 2, 3, 4, 5] \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5}) + \frac{R}{2}e^{i\phi},$$

$$[1, 2, 3, 4, 6] \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_6}) + \frac{R}{2}e^{i\phi},$$

. . . . . .

$$[2, 3, 4, 5, 6] \text{ is } \frac{R}{2}(e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} + e^{i\theta_6}) + \frac{R}{2}e^{i\phi}.$$

These 6 circles have the common point,

$$\frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} + e^{i\theta_6}).$$

In the next step the 7 points, of which $\{1, 2, 3, 4, 5, 6\}$ is one, lie on the circle whose center is $\frac{R}{2}\sum_{p=1}^{p=7} e^{i\theta_p}$ and whose radius is $R/2$. Then follows a group of 8 such circles having the common point, $\frac{R}{2}\sum_{p=1}^{p=8} e^{i\theta_p}$, etc., etc.

In general, with an even number of initial points this method yields a point. The vector to this point from the origin is half the sum of the vectors to the initial points. With an odd number of initial points we have a circle whose radius is half that of the original circle and the vector to whose center is half the sum of the vectors to the initial points. The "W" points on this circle are in perspective relation to the initial points.

CALCULUS.

363. Proposed by B. F. FINKEL, Drury College.

The axis of a right prism whose cross-section is a regular polygon of n sides coincides with the diameter of a sphere of radius R. Find the surface of the sphere included within the prism.

I. Solution by H. S. Uhler, Yale University.

Let a denote the apothem of the right-section of the prism. The required area will be expressed as a function of a, n, and R. This area may be analyzed as $4n$ triangles on the surface of

the sphere. Each triangle is bounded by arcs of two great circles and one small circle. The plane of one great circle contains the axis of the prism and the apothem to the chosen lateral face. The other great circle is determined by the same axis and a lateral edge of the selected face. The small circle is the intersection of the same lateral face and the spherical surface.

An element of surface of the sphere may be written $R^2 \sin\phi \, d\phi d\theta$, where ϕ is the polar angle reckoned from the axis of the prism and θ is the azimuth with respect to the plane containing the apothem. From the geometry of the problem it is readily seen that the total area required is expressed by

$$A = 4nR^2 \int_0^{\pi/n} d\theta \int_0^{\Phi} \sin\phi \, d\phi, \text{ where } \sin\Phi = \frac{a}{R}\sec\theta;$$

whence

$$A = 4nR^2 \int^{\pi/n} \left(1 - \frac{1}{R}\sqrt{R^2 - a^2\sec^2\theta}\right) d\theta$$

or

$$A = 4\pi R^2 - 4nR \int_0^{\pi/n} (\sqrt{R^2 - a^2\sec^2\theta}) d\theta.$$

The indefinite integral may be evaluated as follows. Let $y = \sin\theta$, then

$$\int (\sqrt{R^2 - a^2\sec^2\theta}) d\theta = \int \frac{(\sqrt{(R^2 - a^2) - R^2y^2}) dy}{1 - y^2}$$

$$= \frac{1}{2}\int \frac{(\sqrt{(R^2 - a^2) - R^2y^2}) dy}{1 + y} + \frac{1}{2}\int \frac{(\sqrt{(R^2 - a^2) - R^2y^2}) dy}{1 - y}.$$

The integral

$$U = \int \frac{(\sqrt{\alpha^2 - y^2}) dy}{1 + y} = \int \frac{(\sqrt{\alpha^2 - 1 + 2z - z^2}) dz}{z} \quad \text{if} \quad z = 1 + y.$$

Formula 187 of B. O. Peirce's *A Short Table of Integrals* gives

$$U = \sqrt{\alpha^2 - 1 + 2z - z^2} + \int \frac{dz}{\sqrt{\alpha^2 - 1 + 2z - z^2}} - (1 - \alpha^2)\int \frac{dz}{z\sqrt{\alpha^2 - 1 + 2z - z^2}}.$$

The first and second of these integrals can be reduced by formulae 161 and 183 respectively, so that

$$U = \sqrt{\alpha^2 - 1 + 2z - z^2} + \sin^{-1}\left(\frac{z - 1}{\alpha}\right) - (\sqrt{1 - \alpha^2}) \sin^{-1}\left(\frac{z + \alpha^2 - 1}{\alpha z}\right).$$

In like manner, the substitution $u = 1 - y$ gives

$$\int \frac{(\sqrt{\alpha^2 - y^2}) dy}{1 - y} = -\sqrt{\alpha^2 - 1 + 2u - u^2} - \sin^{-1}\left(\frac{u - 1}{\alpha}\right) + (\sqrt{1 - \alpha^2}) \sin^{-1}\left(\frac{u + \alpha^2 - 1}{\alpha u}\right).$$

Returning to θ and noting that $\alpha = (1/R)\sqrt{R^2 - a^2}$ we find

$$\int (\sqrt{R^2 - a^2\sec^2\theta}) d\theta = R \sin^{-1}\left(\frac{R\sin\theta}{\sqrt{R^2 - a^2}}\right)$$

$$+ \frac{a}{2}\sin^{-1}\left[\frac{R^2(1 - \sin\theta) - a^2}{R\sqrt{R^2 - a^2}(1 - \sin\theta)}\right] - \frac{a}{2}\sin^{-1}\left[\frac{R^2(1 + \sin\theta) - a^2}{R\sqrt{R^2 - a^2}(1 + \sin\theta)}\right].$$

Making use of the relation $\sin^{-1}\beta - \sin^{-1}\gamma = \sin^{-1}(\beta\sqrt{1 - \gamma^2} - \gamma\sqrt{1 - \beta^2})$ the following form (which I have verified by differentiation) is obtained, namely

$$\int (\sqrt{R^2 - a^2\sec^2\theta}) d\theta = R \sin^{-1}\left(\frac{R\sin\theta}{\sqrt{R^2 - a^2}}\right) - \frac{a}{2}\sin^{-1}\left[\frac{2a(\tan\theta)(\sqrt{R^2 - a^2\sec^2\theta})}{R^2 - a^2}\right].$$

Finally $\infty > n \geqq 3$

$$A = 4\pi R^2 - 4nR^2 \sin^{-1}\left(\frac{R\sin\frac{\pi}{n}}{\sqrt{R^2 - a^2}}\right) + 2anR \sin^{-1}\left[\frac{2a\left(\tan\frac{\pi}{n}\right)\left(\sqrt{R^2 - a^2\sec^2\frac{\pi}{n}}\right)}{R^2 - a^2}\right].$$

REMARKS. In the preceding analysis it has been tacitly assumed that we are dealing with the general case where the small circles intersect in real points thus forming a scalloped polar zone. Keeping n and R constant and increasing a, from a sufficiently small value, a time will come when the small circles will be externally tangent to one another. This will happen when the lateral edges of the prism are tangent to the sphere, that is, when the cross-section of the prism becomes inscribed in a *great* circle. Then $a = R\cos(\pi/n)$. This equation also follows at once from the second term of the final formula for A since the greatest value attainable by the sine of an angle is unity. In other words, the equation

$$\frac{R\sin(\pi/n)}{\sqrt{R^2 - a^2}} = 1$$

leads to $a = R\cos(\pi/n)$. Again, this relation affords a means of checking the formula for A. Substituting $R\cos(\pi/n)$ for a in function A and taking π for the value of $\sin^{-1} 0$, we find

$$A = 4\pi R^2 - 2\pi nR^2 + 2\pi nR^2\cos(\pi/n) = 4\pi R^2(1 - n\sin^2(\pi/2n)).$$

Precisely the same result is obtained from the theorem that the area of a zone of one base is equal to the product of its altitude $(R - R\cos(\pi/n))$ by the circumference of a great circle. When a starts from zero and increases to

$$\frac{R\cos(\pi/n)}{\sqrt{1 + \sin^2(\pi/n)}}$$

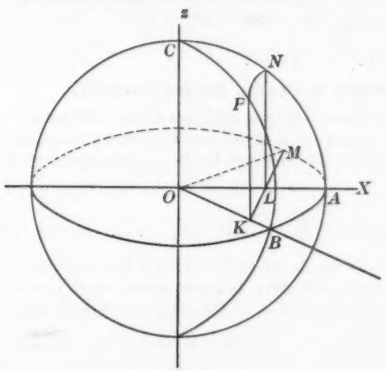
the expression in brackets in the last term of the formula for A increases from 0 to 1. Then as a continues to increase up to $R\cos(\pi/n)$ the expression between the brackets decreases from 1 to 0. Therefore the angle increases continuously from 0 to π. When a exceeds $R\cos(\pi/n)$ the small circles do not intersect, the formula for A does not apply, and the problem degenerates to the calculation of areas of zones of one base. It may also be remarked that the formula for A can be rationalized by making use of the relation $2\sin^{-1} x = \cos^{-1}(1 - 2x^2)$.

A very convincing check on the correctness of the formula for A is afforded by making n infinite while keeping a and R constant. We should then find $A = 4\pi R^2 - 4\pi R(R^2 - a^2)^{\frac{1}{2}}$ for the sum of the areas of the two zones cut out by a coaxial cylinder of radius a. By expanding in infinite series the functions involving n and then substituting $n = \infty$ we obtain at once the limits $-4\pi R^3(R^2 - a^2)^{-\frac{1}{2}}$ and $+4\pi a^2R(R^2 - a^2)^{-\frac{1}{2}}$ for the second and third terms of the expression for A, respectively. But $-4\pi R^3(R^2 - a^2)^{-\frac{1}{2}} + 4\pi a^2R(R^2 - a^2)^{-\frac{1}{2}} = -4\pi R(R^2 - a^2)^{\frac{1}{2}}$, as required.

II. SOLUTION BY THE PROPOSER.

Let O, the center of the given sphere, be the origin of Cartesian coördinates; the axis OC, of the prism, the z-axis; the line OX through L, the mid-point of a side of the regular polygon formed by the intersection of the prism and a plane through O perpendicular to the axis of the prism, the x-axis; and OY, $\perp OA$ and OC, the y-axis. Call OL, the apothem of this polygon, b.

The equation of the sphere is then $x^2 + y^2 + z^2 = R^2$ and the equation of the line OB is $y = x\tan(\pi/n) = mx$, say. Then the surface of the sphere included by the prism is $4n$ times the area of the triangle PCN. The required surface is, therefore,



$$S = 4n\int_0^b\int_0^{mx}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy = 4n\int_0^b\int_0^{mx}\frac{Rdxdy}{\sqrt{R^2-x^2-y^2}}$$

$$= 4nR\int_0^b \sin^{-1}\frac{y}{\sqrt{R^2-x^2}}\Big]_0^{mx}dx = 4nR\int_0^b \sin^{-1}\left(\frac{mx}{\sqrt{R^2-x^2}}\right)dx,$$

$$= 4nR\left[x\sin^{-1}\frac{mx}{\sqrt{R^2-x^2}} - mR\int\frac{xdx}{(R^2-x^2)\sqrt{R^2-(1+m^2)x^2}}\right]_0^b,$$

by integrating by parts.

To integrate the last expression, let $x = R\sin\theta/\sqrt{1+m^2}$. The integral then becomes

$$-mR\int\frac{\sin\theta d\theta}{1+m^2-\sin^2\theta} = mR\int\frac{-\sin\theta d\theta}{m^2+\cos^2\theta} = R\tan^{-1}\left(\frac{\cos\theta}{m}\right) = R\tan^{-1}\left[\frac{\sqrt{R^2-(1+m^2)x^2}}{mR}\right].$$

Hence,

$$S = 4nR\left[x\sin^{-1}\frac{mx}{\sqrt{R^2-x^2}} + R\tan^{-1}\frac{\sqrt{R^2-(1+m^2)x^2}}{mR}\right]_0^b$$

$$= 4nR\left\{b\sin^{-1}\left(\frac{mb}{\sqrt{R^2-b^2}}\right) - R\left[\tan^{-1}\left(\frac{1}{m}\right) - \tan^{-1}\frac{\sqrt{R^2-(1+m^2)b^2}}{\sqrt{R^2-b^2}}\right]\right\},$$

$$= 4nR\left\{b\sin^{-1}\frac{mb}{\sqrt{R^2-b^2}} - R\sin^{-1}\left[\frac{m}{1+m^2}\left(\frac{R-\sqrt{R^2-(1+m^2)b^2}}{\sqrt{R^2-b^2}}\right)\right]\right\},$$

$$= 4nR\left\{b\sin^{-1}\tan\left(\frac{\pi}{n}\cdot\frac{b}{\sqrt{R^2-b^2}}\right) - R\sin^{-1}\left[\tfrac{1}{2}\sin\frac{2\pi}{n}\cdot\frac{R-\sqrt{R^2-b^2\sec^2\frac{\pi}{n}}}{\sqrt{R^2-b^2}}\right]\right\}.$$

When $n = 4$,

$$S = 8R\left[2b\sin^{-1}\frac{b}{\sqrt{R^2-b^2}} - 2R\sin^{-1}\left(\frac{1}{2}\cdot\frac{R-\sqrt{R^2-2b^2}}{\sqrt{R^2-b^2}}\right)\right],$$

$$= 8R\left[2b\sin^{-1}\frac{b}{\sqrt{R^2-b^2}} - R\sin^{-1}\left(\frac{b^2}{R^2-b^2}\right)\right],$$

a result agreeing with that in Osborne's *An Elementary Treatise on the Differential and Integral Calculus*, p. 270, example 5. When $n \doteq \infty$, the prism $\doteq$ a cylinder and

$$S \doteq 4R\left[\frac{\pi b^2}{\sqrt{R^2-b^2}} - \pi R\left(\frac{R-\sqrt{R^2-b^2}}{\sqrt{R^2-b^2}}\right)\right] = 4\pi R\left[\frac{b^2-R^2}{\sqrt{R^2-b^2}} + R\right] = 4\pi R(R-\sqrt{R^2-b^2}),$$

that is, twice the circumference of a great circle of the sphere times the altitude of the zone, or the surface of two zones of altitude $(R-\sqrt{R^2-b^2})$.

Also solved by J. W. CLAWSON.

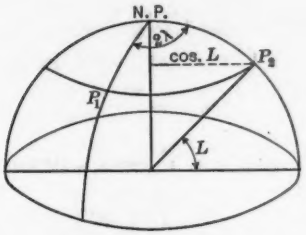
381. Proposed by ELBERT H. CLARK, Purdue University.

Of all points having the same latitude and a constant difference α in their longitudes, to find the latitude of the two so situated that the distance between them, measured along their common parallel of latitude, shall exceed the distance between them measured on their great circle by the greatest possible amount.

SOLUTION BY W. C. EELLS, U. S. Naval Academy.

Let L be latitude of the points, 2λ (in radians) their difference in longitude, $2d$ and $2D$ the small circle latitude and great circle distances between them, and $y = d - D$. Let the earth be a sphere of radius unity. Since the radius of the latitude circle is $\cos L$,

$$d = \lambda\cos L. \qquad (1)$$



Dropping a perpendicular from the north pole to the great circle through the two points, we have two right triangles. Whence, by Napier's rules of circular parts,

$$\sin D = \sin \lambda \cos L. \tag{2}$$

From (1) and (2), $y = \lambda \cos L - \text{arc sin } (\sin \lambda \cos L)$, and it is desired to find the maximum value of y.

Let

$$\frac{dy}{dL} = -\lambda \sin L + \frac{\sin L \sin \lambda}{\sqrt{1 - \sin^2 \lambda \cos^2 L}} = 0.$$

Then, $\sin L = 0$ is a solution, giving $L = 0$, obviously the minimum value of y.

The other solution is

$$\cos L = \pm \frac{\sqrt{\lambda^2 - \sin^2 \lambda}}{\lambda \sin \lambda},$$

the maximum. Whence, L can be found for any given λ. As λ varies from 0° to 90°, 2λ varies from 0° to 180° and L varies from 90° to 39° 32′ 23″. For $\lambda = 30°$, $L = 53°\ 34'\ 42''$ and for $\lambda = 60°$, $L = 49°\ 31'\ 16''$.

Also solved by J. A. Caparo, Paul Capron, Horace Olson, C. N. Schmall.

382. Proposed by B. J. BROWN, Student in Drury College.

Discuss for what values of m and n, the integral, $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, is finite and show how this integral can be expressed by means of integrals of the form $\int_0^\infty e^{-x}x^{p-1}dx$.

Solution by H. L. Agard, Williams College.

1°. The integrand has no singular points for $m \geqq 1$ and $n \geqq 1$. The only singular points are $x = 0$ for $m < 1$ and $x = 1$ for $n < 1$.

Pierpont[1] states the

Theorem. Let $f(x)$ be regular in the interval (a, b), except at a. For some $0 < \mu < 1$, let $R \lim_{x=a} (x-a)^\mu |f(x)|$ be finite. Then $f(x)$ is absolutely integrable in (a, b).

[The symbol $R \lim_{x=a}$ indicates that $x \doteq a$ from the right. A similar meaning attaches to $L \lim_{x=a}$. The theorem applies equally to a singularity at b.]

Consider

$$R \lim_{x=0} x^\mu |x^{m-1}(1-x)^{n-1}|, \quad m < 1.$$

Evidently, this limit is finite for $m > 0$ and $1 - m < \mu < 1$.

Consider

$$L \lim_{x=1} (1-x)^\mu |x^{m-1}(1-x)^{n-1}|, \quad n < 1.$$

Evidently, this limit is finite for $n > 0$ and $1 - n < \mu < 1$.

Hence, by the above theorem, the integral is absolutely convergent if both $m > 0$ and $n > 0$.

[1] Pierpont, *Theory of Functions of Real Variables*, Vol. I, p. 407.

2°. The integral under consideration is known as the Beta function, or first Eulerian integral, and is denoted by $B(m, n)$. This function is to be expressed in terms of the Gamma function, or second Eulerian integral, $\Gamma(m) = \int_0^\infty e^{-x}x^{m-1}dx$. The required relation is

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

In

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \tag{1}$$

set $x = y/(1+y)$, obtaining,

$$B(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy. \tag{2}$$

In

$$\Gamma(m) = \int_0^\infty e^{-z}z^{m-1}dz,$$

if we set $z = ax$, we get the formula,

$$\frac{1}{a^m} = \frac{1}{\Gamma(m)}\int_0^\infty e^{-ax}x^{m-1}dx. \tag{3}$$

In (3) replace a by $1+y$ and m by $m+n$, obtaining,

$$\frac{1}{(1+y)^{m+n}} = \frac{1}{\Gamma(m+n)}\int_0^\infty e^{-(1+y)x}x^{m+n-1}dx. \tag{4}$$

(2) and (4) give

$$B(m, n) = \frac{1}{\Gamma(m+n)}\int_0^\infty dy\int_0^\infty e^{-(1+y)x}x^{m+n-1}y^{m-1}dx. \tag{5}$$

In (5) the order of integration may be inverted.[2] For, (1) the integrand, $f(x, y)$, is continuous in the first quadrant, except on the axes; (2) the integrals, $\int_0^\infty f(x, y)dx$ and $\int_0^\infty f(x, y)dy$ are uniformly convergent[3] in any intervals (a, b), $a>0$, and (α, β), $\alpha>0$, respectively; and (3) $\int_0^\infty dx\int_0^\infty f(x, y)dy$ exists, since

$$\begin{aligned}\int_0^\infty f(x, y)dy &= e^{-x}x^{m+n-1}\int_0^\infty e^{-xy}y^{m-1}dy \\ &= e^{-x}x^{m+n-1}\cdot x^{-m}\Gamma(m), \text{ by (3)} \\ &= e^{-x}x^{n-1}\Gamma(m) \end{aligned} \tag{6}$$

and, therefore, for $a>0$, using (6),

$$\int_0^\infty dx\int_0^\infty f(x, y)dy = \Gamma(m)\lim_{a=0}\int_a^\infty e^{-x}x^{n-1}dx = \Gamma(m)\Gamma(n). \tag{7}$$

The conditions for inversion being fulfilled, (5) and (7) give the desired result,

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

383. Proposed by WILLIAM CULLUM, Albion, Mich.

Find the area of the curved surface of a right cone whose base is the asteroid, $x^{2/3}+y^{2/3}=a^{2/3}$, and whose altitude is h. From Townsend and Goodenough's *First Course in Calculus*, p. 288, Ex. 11.

Note. Among other methods, find the required area by means of the formula

$$\iint\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dydx.$$

EDITORS.

[2] See Nielson's *Handbuch d. Theorie d. Gamma Functionen*, pages 148–9. Pierpont, *loc. cit.*, §680, 3.

[3] Pierpont, *loc. cit.*, § 663.

Two Solutions by F. L. Griffin, Reed College.

(I) *By a single integration.* The element of area is a triangle, whose base ds is an elementary arc of the asteroid, and whose altitude H is the perpendicular from the vertex of the cone upon the tangent to the arc ds. This perpendicular is the hypotenuse of a right triangle whose legs are the height of the cone h, and the perpendicular p from the center of the base upon the tangent. Hence

$$dS = \tfrac{1}{2}Hds, \qquad \text{where} \qquad H = \sqrt{h^2 + p^2}.$$

From the parametric equations of the asteroid $x = a\cos^3 t$, $y = a\sin^3 t$, we find $ds = \sqrt{dx^2 + dy^2} = \frac{3}{2}a\sin 2t dt$. Also, since $dy/dx = -\tan t$, the equation of the tangent line is $Y - a\sin^3 t = -\tan t(X - a\cos^3 t)$. On reducing this to the normal form we find

$$p = \frac{a\sin^3 t + \tan t \cdot a\cos^3 t}{\sec t} = \tfrac{1}{2}a\sin 2t;$$

whence

$$H = \sqrt{h^2 + \frac{a^2}{4}\sin^2 2t} = \tfrac{1}{2}\sqrt{(4h^2 + a^2) - a^2\cos^2 2t}.$$

Finally,

$$S = \frac{3a}{2}\int_0^{\pi/2} \sqrt{(4h^2 + a^2) - a^2\cos^2 2t}\cdot \sin 2t dt,$$

which reduces, on putting $\cos 2t = z$, to

$$S = \tfrac{3}{4}a^2\int_{-1}^{1}\sqrt{\frac{4h^2 + a^2}{a^2} - z^2}dz = \tfrac{3}{2}ah + \tfrac{3}{4}(4h^2 + a^2)\sin^{-1}\frac{a}{\sqrt{4h^2 + a^2}}.$$

[This result, by the way, has a simple interpretation: Three times the right triangle which forms half of the broadest vertical section of the cone, increased by three times a circular sector whose radius is the smallest slant height of the cone and whose angle is the angle between two such opposite smallest slant heights.]

(II) *By the formula* $\int\int \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}dydx$. The equation of the surface, taking the origin at the center of the base, is

$$z = h - \frac{h}{a}(x^{2/3} + y^{2/3})^{3/2}$$

whence

$$S = 4\int_0^a \int_0^{(a^{2/3} - x^{2/3})^{3/2}} \sqrt{x^{2/3}y^{2/3} + \frac{h^2}{a^2}(x^{2/3} + y^{2/3})^2 x^{-1/3}y^{-1/3}}dydx.$$

Put $x = au^{3/2}$, $y = av^{3/2}$, $h/a = 1/k$, and we have

$$S = \frac{9a^2}{k}\int_0^1\int_0^{1-u}\sqrt{k^2uv + (u + v)^2}dvdu$$

$$= \frac{9a^2}{k}\int_0^1 du\left[\frac{2v + (2 + k^2)u}{8}\sqrt{(2v + 2 + k^2u)^2 - (k^4 + 4k^2)u^2}\right.$$

$$\left. - \frac{(k^4 + 4k^2)u^2}{8}\log\left(\frac{2v + 2 + k^2u + \sqrt{(2v + 2 + k^2u)^2 - (k^4 + 4k^2)u^2}}{2}\right)\right]_0^{1-u}$$

$$= I_1 - I_2 - I_3 + I_4,$$

where

$$I_1 = \frac{9a^2}{4k}\int_0^1 (2 + k^2u)\sqrt{1 + k^2u - k^2u^2}du, \qquad I_2 = \frac{9a^2(k^2 + 2)}{4k}\int_0^1 u^2du = \frac{3a^2(k^2 + 2)}{4k},$$

$$I_3 = \frac{9a^2k(k^2 + 4)}{8}\int_0^1 u^2\log(2 + k^2u + 2\sqrt{1 + k^2u - k^2u^2})du,$$

$$I_4 = \frac{9a^2k(k^2 + 4)}{8}\int_0^1 u^2[\log u + \log(k^2 + 4)]du = \frac{a^2k(k^2 + 4)}{8}[\log(k^2 + 4)^3 - 1].$$

To evaluate I_1, complete the square and put

$$u = \frac{1}{2}\left(1 + \frac{\sqrt{k^2+4}}{k}\sin\theta\right).$$

Then

$$\sqrt{1 + k^2u - k^2u^2} = \frac{\sqrt{k^2+4}}{2}\cos\theta,$$

and $2 + k^2u = \frac{1}{2}(k^2+4)(1 + \sin\alpha\sin\theta)$ where $\alpha = \sin^{-1}(k/\sqrt{k^2+4}) = \cos^{-1}(2/\sqrt{k^2+4})$. Hence,

$$I_1 = \frac{9a^2(k^2+4)^2}{32k^2}\int_{-\alpha}^{\alpha}(1 + \sin\alpha\sin\theta)\cos^3\theta d\theta$$

$$= \frac{9a^2(k^2+4)^2}{32k^2}\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta - \tfrac{1}{3}\sin\alpha\cos^3\theta\right]_{-\alpha}^{\alpha} = \frac{9a^2(k^2+4)^2}{32k^2}\left[\alpha + \frac{2k}{k^2+4}\right].$$

To evalute I_2, first integrate by parts, putting

$$U = \log(2 + k^2u + 2\sqrt{1 + k^2u - k^2u^2}), \qquad dV = \frac{9a^2k(k^2+4)}{8}u^2du.$$

Then

$$dU = \frac{k^2du}{2R}\left\{1 - \frac{(k^2+4)u}{2 + k^2u + 2R}\right\}, \qquad V = \frac{3a^2k(k^2+4)}{8}u^3,$$

where $R = \sqrt{1 + k^2u - k^2u^2}$. Then

$$I_2 = \frac{3a^2k(k^2+4)}{8}\left\{\left[u^3\log(2 + k^2u + 2R)\right]_0^1 - \frac{k^2}{2}\int_0^1\frac{u^3du}{R} + \frac{k^2(k^2+4)}{2}\int_0^1\frac{u^4du}{R(2 + k^2u + 2R)}\right\}$$

$$= \frac{3a^2k(k^2+4)}{8}\log(k^2+4) - I_3 + I_6,$$

where

$$I_3 = \frac{3a^2k^3(k^2+4)}{16}\int_0^1\frac{u^3du}{\sqrt{1 + k^2u - k^2u^2}} = \frac{3a^2(k^2+4)^{\frac{3}{2}}}{128k}\int_{-\alpha}^{\alpha}(\sin\alpha + \sin\theta)^3d\theta,$$

$$= \frac{3a^2(k^2+4)^{\frac{3}{2}}}{128k}\left[2\alpha\sin^3\alpha + 3\alpha\sin\alpha - 3\sin^2\alpha\cos\alpha\right],$$

$$= \frac{3a^2(k^2+4)}{128}\left[(5k^2+12)\alpha - 6k\right],$$

and

$$I_6 = \frac{3a^2k^3(k^2+4)^2}{16}\int_0^1\frac{u^4du}{\sqrt{1 + k^2u - k^2u^2}(2 + k^2u + 2\sqrt{1 + k^2u - k^2u^2})}$$

$$= \frac{3a^2(k^2+4)^3}{128k^2}\int_{-\alpha}^{\alpha}\frac{(\sin\alpha + \sin\theta)^4d\theta}{\sin\alpha\sin\theta + \cos\alpha\cos\theta + 1},$$

since

$$2 + k^2u = \tfrac{1}{2}(k^2+4)(1 + \sin\alpha\sin\theta) \qquad \text{and} \qquad 2R = \sqrt{k^2+4}\cos\theta = \tfrac{1}{2}(k^2+4)\cos\alpha\cos\theta.$$

Hence,

$$I_6 = \frac{3a^2(k^2+4)^3}{128k^2}\int_{-\alpha}^{\alpha}\frac{\left(2\sin\frac{\theta+\alpha}{2}\cos\frac{\theta-\alpha}{2}\right)^4d\theta}{2\cos^2\frac{\theta-\alpha}{2}}$$

$$= \frac{3a^2(k^2+4)^3}{16k^2}\int_{-\alpha}^{\alpha}\sin^4\left(\frac{\theta+\alpha}{2}\right)\cos^2\frac{\theta-\alpha}{2}d\theta.$$

By making imaginary exponential substitutions, or putting $\theta = 2\phi + \alpha$ and expanding, we find

$$I_6 = \frac{3a^2(k^2+4)^3}{512k^2}[\sin(2\theta+2\alpha) - 8\sin(\theta+\alpha) + 6\theta + \tfrac{1}{3}\sin(3\theta+\alpha) + \sin(\theta+3\alpha) - 2\sin 2\theta - 4\theta\cos 2\alpha + 6\sin(\theta-\alpha)]_\alpha^\pi$$

$$= \frac{3a^2(k^2+4)^3}{128k^2}[\tfrac{1}{3}\sin\alpha\cos^3\alpha - \tfrac{1}{3}\sin\alpha\cos\alpha + \alpha(1+4\sin^2\alpha)]$$

$$= \frac{a^2(k^2+4)}{128k^2}[-2k(17k^2+12) + (15k^2+12)(k^2+4)\alpha].$$

Collecting results we find, since $S = I_1 - I_2 - I_3 + I_4$,

$$S = \alpha\left\{\frac{9a^2(k^2+4)^2}{32k^2} + \frac{3a^2(k^2+4)(5k^2+12)}{128} - \frac{a^2(k^2+4)^2(15k^2+12)}{128k^2}\right\}$$

$$+ \frac{a^2}{64k}\{36(k^2+4) - 48(k^2+2) - 8k^2(k^2+4) - 9k^2(k^2+4) + (17k^2+12)(k^2+4)\}$$

$$= \frac{3}{2}\frac{a^2}{k} + \tfrac{3}{4}a^2\frac{k^2+4}{k^2}\alpha = \tfrac{3}{2}ah + \tfrac{3}{4}(4h^2+a^2)\sin^{-1}\frac{a}{\sqrt{4h^2+a^2}}.$$

Also solved by G. W. HARTWELL and the PROPOSER.

398A. Proposed by EMMA GIBSON, Drury College.

[This problem was unnumbered in the December issue.]

Solve the differential equation

$$(x^3y^3 + x^2y^2 + xy + 1)y + (x^3y^3 - x^2y^2 - xy + 1)x\frac{dy}{dx} = 0.$$

SOLUTION BY FREDERICK WOOD, University of Wisconsin.

The equation

$$(x^3y^3 + x^2y^2 + xy + 1)y + (x^3y^3 - x^2y^2 - xy + 1)x\frac{dy}{dx} = 0$$

reduces to

$$(x^3y^3 + 1)(ydx + xdy) + (x^2y^2 + xy)(ydx - xdy) = 0$$

and finally to

$$x^2y^2(ydx + xdy) + ydx + xdy - 2x^2ydy = 0$$

from which we get

$$ydx + xdy + \frac{ydx + xdy}{x^2y^2} - \frac{2dy}{y} = 0.$$

This becomes, after integration,

$$xy - \frac{1}{xy} - 2\log y = c, \qquad \text{or} \qquad x^2y^2 - 2xy\log c'y = 1.$$

Also solved by H. POLISH, W. W. BEMAN, H. S. UHLER, ELIJAH SWIFT, HORACE OLSON, S. E. RASOR, G. KEULIGAN, H. L. AGARD, ELMER SCHUYLER, and B. J. BROWN.

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

REPLIES.

24. The following facts are significant:

(1) The New England Association of Mathematics Teachers has appointed a committee "to investigate the current criticisms of high school mathematics."

(2) A committee of the Council of the American Mathematical Society has under consideration the question "whether any action is desirable on the part of the Society in the matter of the movement against mathematics in the schools."

(3) At the recent meeting in Cincinnati of the National Education Association an iconoclastic discussion on the topic: "Can algebra and geometry be reorganized so as to justify their

retention for high school pupils not likely to enter technical schools?" aroused approbation and applause. An outline of the remarks by one of the speakers was printed in a previous issue.

In view of these facts what should be done by those who believe in the value of mathematics as a general high school study?

REPLY BY CHARLES N. MOORE, University of Cincinnati.

The best thing to do is to devote a little time and effort to pointing out the essential unsoundness of the arguments that have been advanced against the retention of mathematics as a required subject in the high schools. The movement against mathematics is, for the most part, confined to a group of educational theorists who feel that they must advocate something new in order to convince their readers that they are investigators. This group, however, has made up in volume of sound for what it has lacked in numbers and, in consequence, has deceived many people into thinking that it represents a widespread trend of thought.

The statement that there is at the present time much uncertainty as to the educational value of algebra and geometry will not bear examination. That the thinking public is as firmly convinced as ever of the high educational value of these subjects was conclusively shown by the answers to a questionnaire sent out recently by Professor Hancock. This questionnaire was sent to a group of the most prominent physicians, clergymen, lawyers and business men throughout the country; it was also sent to a similar group of residents of Cincinnati. In the first group 90 out of a total of 99 advocated a high school course in which mathematics was a required subject; in the second group 96 out of a total of 105 did the same thing.[1] In view of these replies it seems to the writer that the burden of proof rests upon those who wish to displace mathematics as a required subject for high school pupils.

The statement that recent investigations have thrown doubt on the disciplinary value of mathematical study is absolutely without justification. The chief disciplinary value of mathematics is in the training of the reasoning powers it affords. The writer has gone over the literature on transfer of training quite recently, and does not know of any experiments that involved training in logical reasoning. It is absurd to contend that experiments based on the marking of certain letters on a printed page or guessing the size of pieces of paper, will enable one to draw valid conclusions with regard to the training afforded by the study of mathematics.

The statement that the value of mathematics as an instrument has been greatly exaggerated, is also without foundation. As a matter of fact the majority of people do not realize the wide extent of the applications of mathematics. The writer is continually meeting people who are deploring their lack of mathematical training because their deficiency in this respect renders it difficult or impossible to pursue some particular study or calling in which they are interested. In a recent article in *Science*,[2] Professor S. G. Barton pointed out the fact that in

[1] For a more detailed account of Professor Hancock's investigation, *cf. School and Society*, Vol. I (1915), p. 893.

[2] Volume 40 (1914), page 697.

104 articles of the Encyclopedia Britannica, eleventh edition, use was made of the symbols of the infinitesimal calculus. Only about one fourth of these articles could be classed as dealing with pure mathematics.

The statement that the great majority of the teachers of mathematical subjects in high schools are not informed as to "the ultimate educational values" of these studies, seems much exaggerated. Undoubtedly some of these teachers are not so informed, and the reason for it lies in the deficiency of their own mathematical training. The remedy for this lies in demanding a greater amount of preliminary study of mathematics as a qualification for the teaching of that subject in the high schools.

The suggestion that mathematics might be made more cultural by introducing purely descriptive courses in this subject, is in line with the effort in certain quarters to convert our high schools into kindergartens for students of high school age. The idea that mathematics could be "interpreted" to a student that never had a mathematical idea in his head, would be amusing if it were not advocated as a serious proposal.

DISCUSSIONS.

RELATING TO THE "SIMSON LINE" OR "WALLACE LINE."

By Roger A. Johnson, Western Reserve University.

(Cf. statement in this Monthly for June, 1915, p. 201, problem No. 467.)

It is a well-known theorem that the feet of the perpendiculars from a point to the sides of a triangle are collinear if and only if the point lies on the circumscribed circle of the triangle. This theorem has always been known as Simson's, and the line as Simson's line, until recent times, when an attempt has been made to replace Simson's name by that of Wallace. See, for instance, this Monthly, June, 1915, p. 201, problem 467. The purpose of this note is to submit the proposition that the name "Simson Line" should be retained. It would be desirable, perhaps, to have the views of readers of the Monthly on the matter.

Let us first determine the facts. Some thirty years ago, the inquiry was made as to where in Simson's works the theorem in question was to be found. Search was made, especially by that able investigator J. S. Mackay of Edinburgh, who read all known writings of Simson, without finding an allusion to the theorem. Further investigation persuaded him that the fact was discovered by one William Wallace about 1798; it seems to have appeared in print for the first time in that year, in Leybourn's Mathematical Repository.[1] Mackay also[2] traces the source of the erroneous ascription to Simson. Early in the century, Servois[3] cites the theorem, introducing it in the following words, "le théorème suivant, qui est, je crois, de Simson." A little later, Poncelet in his "Propriétés Projectives," 468, reproduced this statement without the qualification, and his authority was evidently suf-

[1] Cf. Muir, *Proceedings of the Edinburgh Math. Society*, Vol. 3, p. 104.
[2] Mackay, *Proceedings of the Edinburgh Math. Society*, Vol. 9, pp. 83–91.
[3] Gergonne's *Annales de Math.*, Vol. IV, p. 250.

ficient to perpetuate the error. Certainly, until the last decade of the nineteenth century, no one doubted that the theorem was Simson's. Now, however, there seems no reason to believe that he ever heard of it.

If, then, the theorem is Wallace's, should it not be given his name, in tardy recognition of the debt we owe him? After Mackay's researches had been published, he and other Scotch writers made the innovation, unwisely it seems to the present writer. At present, custom is divided. No author dares use one name without mentioning the other.

In any such situation, it is obvious that an attempt to effect a change from a universally established term to a new one will involve such an amount of confusion that we must convince ourselves that the advantages are sufficient to outweigh the inconveniences. The most powerful, and perhaps only reason for changing is that of justice, to assure to every discoverer the credit due him. This seems inadequate. In general, the fact that a theorem is named for a man is neither a sufficient indication that the theorem was discovered by him, nor, on the other hand, a definite characterization of the theorem. The classic example of an unjustly named theorem is Cardan's solution of the cubic, discovered, as is well known, by Tartaglia; several cases of the same kind will occur to the reader. On the other hand, the expression " Steiner's Theorem," for instance, may mean any of numerous familiar theorems. If, in particular, we continue to designate the line under discussion as Simson's line, or rather as the Simson line, we shall not be implying that it was discovered by Robert Simson, but that his name is attached to it by long-established custom.

There is, however, another reason for opposing the change, more cogent than this. The term " Wallace's line " already has a well defined meaning in another field of science, namely Zoögeography. According to Webster's New International Dictionary, it was named for Dr. Alfred Russel Wallace, and is " an imaginary line separating the Oriental and Australian regions. . . . The faunas on either side of the line are remarkably distinct." Now it is not likely that this line will be confused with the one mentioned in our theorem, but it would seem that since we are in a position to choose, the wisest choice is that which avoids all possibility of confusion.

CORRESPONDENCE.

To the Board of Editors:

Thinking in retrospect of the organization meeting at which the Mathematical Association of America was launched, I desire to congratulate you upon the very successful issues of the meeting, and upon the very auspicious beginning of the new Association.

One thought has occurred to me that I have not yet heard expressed. It relates to the toast on " The economic value of the young " to which Professor Hedrick so aptly responded at the luncheon tendered by the mathematics faculty of the University of Ohio. Now, economic considerations are of very vital import

to the average teacher of mathematics. His salary, as well as his rank and professional standing, are very closely related to the way in which he is regarded by other mathematicians, in his own institution and throughout the country. His advancement usually depends upon a change from one institution to another, or on the prospect of such a change. Until the present time the only way of claiming the regard and good opinion of mathematicians outside of one's own institution has been through research. This has resulted in great emphasis on activities in research and small emphasis on effective teaching, when appointments and promotions are considered. The new association should become an avenue through which an effective teacher may gain the good opinion of the profession at large because of the quality of his work and through contributions to the solution of the problems of the teacher. In this way salary, rank, etc., may come to be more closely related to service rendered. May not the new born Association thus become a very material economic asset to many worthy teachers of mathematics?

The economic motive may become an influence of considerable proportions in promoting activities in the new Association, as I believe it has been in promoting research which finds its expression through the old Society.

RALPH E. ROOT.

U. S. NAVAL ACADEMY,
January 4, 1916.

ECHOES FROM THE COLUMBUS MEETING.

All will be interested to know that the membership list of the Association is growing apace, over one hundred per week being the record for the time since the announcements were distributed. It is taken for granted that all who signed the call for the organization meeting will come in as charter members. In fact, the membership committee of the Council has already elected all of these persons to membership so that there will be no delay when their names are received. Large numbers of them are already in, as well as many of those whose names were received too late to appear in the call.

Attention is again called to the provision of the Constitution that in the case of all who join the Association before April 1, 1916, the initiation fee is to be waived. Doubtless, many who fully intend to take advantage of this provision may find themselves characterized in the following note just received from a prominent member of the faculty of a great New England University:

> You may have wondered why I did not join the Mathematical Association earlier, if you thought about it at all. It has been negligence and not lack of appreciation for the movement you have had so much to do with. I feel a large personal debt to the MONTHLY, and I believe it will do even more good in the future than in the past.

It is confidently believed that such declarations of allegiance, just at this time, to the Mathematical Association of America, will prove to be as important a service to the cause of mathematics in this country as may be rendered by those who are in any way related to the collegiate field. The establishment of a widely

representative membership of large proportions within a period of sixty days (from the distribution of the announcements) will set a record and create a momentum which should sweep all obstacles from the path.

Institutional memberships are coming in as rapidly as could be expected in view of the time required to make such arrangements. One of the universities on the former subsidy list has shown its confidence in the cause by subscribing for an institutional membership for a period of five years in advance. It is believed that many institutions will consider it an honor to be thus connected with this Association.

A correspondent asks whether a delegate sent by an institutional member to a meeting would be counted as an individual member. This is clearly not contemplated by the Constitution. Institutional members are those who support the Association by the payment of the individual dues, and only the names of such persons will appear in the address list of individual members to be printed later. There will also be printed a list of institutional members, but this list will not be accompanied by the names of delegates, since these will vary from time to time. The names of delegates actually in attendance at any meeting will be printed with the report of that meeting. An individual member may, of course, be delegated by an institution to represent it at any meeting. The two copies of the MONTHLY to be sent to institutional members will in all cases be mailed directly to the library of the institution, and not to individual members of the faculty.

Inquiries are coming in with respect to the formation of state sections of the Association. One of particular interest at this time is from the state of Georgia. It is hoped and believed that many new organizations of this character will be formed as soon as there is time for the plan to become sufficiently well known; and also that many existing organizations will be directed into this channel. Emphasis is laid upon the state as a convenient geographical subdivision for a section, because of the fact that there are already mathematical organizations in many cases in connection with the general state teachers' associations, and these can be readily transformed into sections of the new Association. Moreover, the meetings of such sections can be held at the same time as the general state meetings, thus making it possible for larger numbers to get together, and avoiding unnecessary duplication of places and times of meeting. The Committee on the organization of sections is working out the details and will have a report ready to publish in the March issue.

Suggestions have already been made for starting a library for the Association. For the present it will be in charge of the Secretary, and any who wish to contribute books or journals may send them directly to Professor W. D. CAIRNS, Oberlin, Ohio. The Managing Editor wishes to announce that through the kindness of Professor E. H. MOORE, he has been able to make up a complete set of the MONTHLY. These volumes will be bound and placed in the Library at an early date. This is especially fortunate, since it is almost impossible now to secure complete sets of the MONTHLY. Some publishers have also contributed certain books. A complete list of these will be published soon.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University.

A late volume of the Proceedings of the Indiana Academy of Sciences contains an extended treatment of "Some properties of binomial coefficients," by PROFESSOR A. M. KENYON, of Purdue University.

"The theory of invariants," by PROFESSOR O. E. GLENN, of the University of Pennsylvania, has been published by Ginn and Company.

MISS GERTRUDE I. MCCAIN has been appointed professor of mathematics in the Oxford College for Women at Oxford, Ohio.

John Wiley & Sons announce the publication of a book of 390 pages on the "Theory and Applications of Finite Groups," by PROFESSORS G. A. MILLER, H. F. BLICHFELDT, and L. E. DICKSON.

MR. ALTON MILLER, absent on leave from the University of Michigan, is studying under Professor Segre at the University of Torino. Mr. Miller holds a fellowship in mathematics from Harvard University.

The Wyoming School Journal for January, 1916, contains an article by C. EBIN STROMQUIST, professor of mathematics in the University of Wyoming, on "The correlation of mathematics in grades seven to ten."

"The college teacher's function" is the title of a thoughtful article in *School and Society* for January 15, by Professor ARTHUR J. TODD, University of Minnesota.

"College entrance requirements" is a topic discussed with considerable illumination in *School and Society* for January 8, 1916, by Professor LOUIS W. RAPER, of Pennsylvania State College.

"Robert of Chester's Latin translation of the algebra of Al-Khowarizmi," by PROFESSOR L. C. KARPINSKI, has just appeared from the Macmillan Press. Besides an English translation the work contains an introduction giving a brief history of algebra.

"Mathematics and efficiency" is the title of an article in the January number of *School Science and Mathematics*, by FLETCHER DURELL, Headmaster at the Lawrenceville School, New Jersey. In the same issue, Professor G. A. MILLER writes on "Reform in teaching mathematics."

The major part of the October issue of the *Mathematical Gazette* is devoted to "A study of the life and writings of Colin Maclaurin," by Charles Tweedie,

F.R.S.E., including four full-page plates and a critical estimate of his writings. A condensation of this article will be given in a later issue of the MONTHLY.

The *Bollettino di Bibliografia e Storia delle Scienze Matematiche* continues to arrive with commendable regularity in view of the extraordinary conditions created by the war. The closing number of 1915, No. 4, was published Nov. 20, 1915. Besides numerous reviews, the volume for 1915, Vol. XVII, includes an article by L. GODEAUX on a Belgian mathematician of the sixteenth century; articles by J. H. GRAF on the correspondence between Ludwig Schläfli and Italian mathematicians of his period, including to date letters from D. Chelini and E. Beltrami, and an article on Lingi Forni, a mathematician of Pavia, by G. VIVANTI.

Two articles in *Science* for September 11 and October 22, 1915, by Professor C. N. Moore, of the University of Cincinnati, will be of interest to readers of the MONTHLY. The one is on "Correlation and disciplinary values" and the other on "The coefficient of correlation as a measure of relationship." The former gives the results of some statistical studies with reference to the disciplinary value of certain high school subjects, including algebra and geometry, which are being attacked in many quarters. See also Professor Moore's reply to Question 24 in this issue.

A reprint from the Napier Tercentenary memorial volume, published by the Royal Society of Edinburgh, contains the address by Professor FLORIAN CAJORI on "Algebra in Napier's day and alleged prior inventions of logarithms." The three main conclusions are (1) "That John Napier enjoys the all-important right of priority of publication"; (2) "that Joost Bürgi is entitled to the honor of independent invention"; (3) "that Joost Bürgi constructed his table some time between 1603 and 1611, and that John Napier worked on logarithms probably as early as 1594—that, therefore, Napier began working on logarithms probably much earlier than Bürgi."

In the twelfth volume, series III, 1915, of the *Periodico di Matematica* which completed the thirtieth year of publication with the issue of September, 1915, several articles are found which may be of interest to readers of the MONTHLY. Two articles, to be continued, by G. LAZZERI treat of static moments, moments of inertia, and moments of higher order; two articles by D. KRYJANOVSKY, translated from the Russian into Italian apparently by the author himself, treat of maxima and minima of plane figures; an article by Q. PAOLINA discusses the relations between arithmetic and geometric means, based upon the very ancient proposition that the geometric mean between two quantities is less than the arithmetic mean.

The American Mathematical Society held its twenty-second annual meeting in New York, December 27–28, 1915. There were seventy-two members present at the four sessions and thirty-seven papers were presented. The total member-

ship of the Society is now 736. The number of members attending at least one meeting of the Society or its sections during the year 1915 was 253. The total number of registrations at all meetings during the year was 418. The president of the Society is Professor E. B. Brown, of Yale University, and the two vice-presidents are Professor E. R. Hedrick, of the University of Missouri, and Professor Virgil Snyder, of Cornell University.

In the September–November issue of *L'Enseignement Mathématique* PROFESSOR G. A. MILLER writes an article on "The preparation of mathematics teachers in the United States of America," in which he presents a report of the many changes which the American universities have inaugurated during the last twenty-five years, with a view of providing better facilities for students who are teachers (through summer sessions) or who expect to become teachers (through schools of education). He points out, further, the inadequacy of the number of properly equipped teachers to fill all the available positions, the relatively low but increasing salary scale, the wholesome influence of THE AMERICAN MATHEMATICAL SOCIETY on the teachers of higher mathematics, and the awakening of interest among the teachers of secondary mathematics through their numerous teachers' associations and through two periodicals devoted to their interests.

There is a bill before Congress to make the use of the Centigrade thermometer scale obligatory in all government publications, in the hope of bringing about its adoption for all purposes in place of the Fahrenheit scale. This is a move in a good direction, but it raises again the larger question of the metric system as a whole, and we wonder whether the United States will be the last of the civilized nations to adopt that system. An article in *The Scientific Monthly* for December, 1915, by Dr. Joseph V. Collins, of Stevens Point, Wis., discusses the question under the title: "A metrical tragedy," showing that at least two thirds of a year for every child in the land is wasted in the study of our cumbersome system of weights and measures, and that this waste entails an economic loss of possibly three hundred millions of dollars annually. It is an opportune time for all friends of progress in this direction to act, especially as requests have been made for all available data bearing upon the thermometer phase of the question.

The Chicago Section of the American Mathematical Society held its thirty-sixth regular meeting at Columbus, Ohio, on Thursday, Friday and Saturday, December 30–31, 1915, and January 1, 1916. There were sixty-seven members in attendance and twenty-five papers were read, aside from the retiring address of the chairman, Professor E. J. WILCZYNSKI, of the University of Chicago, who spoke at the joint session with Section A of the American Association on "The historical development and future prospects of differential geometry of plane curves." At the same session Professor H. S. WHITE, of Vassar College, gave his retiring address as Vice-President of Section A on the topic: "Poncelet Polygons." Practically all members of the American Mathematical Society who were present at Columbus

joined in the organization meeting of the THE MATHEMATICAL ASSOCIATION OF AMERICA. The dinner on Thursday evening, in fact, resolved itself largely into a congratulatory occasion in honor of the birth of the new Association. The officers of the Chicago Section for the next two years are: *Chairman*, Professor W. B. FORD, University of Michigan; *Secretary*, Professor ARNOLD DRESDEN, University of Wisconsin; *Third member of program committee*, Professor H. L. RIETZ, University of Illinois.

Please note the following addresses of chairmen of committees to whom communications should be sent on the respective subjects:

Problems Proposed and Solved, to Professor B. F. FINKEL, 1228 Clay St., Springfield, Mo.

Questions and Discussions, to Professor U. G. MITCHELL, University of Kansas, Lawrence, Kan.

Notes and News, to Professor D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Books to be Reviewed, to Professor W. H. BUSSEY, University of Minnesota, Minneapolis, Minn.

General Editorial Correspondence, to Professor H. E. SLAUGHT, Managing Editor, 5548 Kenwood Ave., Chicago, Ill.

Business Correspondence concerning both the MONTHLY and the ASSOCIATION, to Professor W. D. CAIRNS, 55 East Lorain St., Oberlin, Ohio.

IMPORTANT NOTICE TO PRESENT MONTHLY SUBSCRIBERS.

Henceforth, the subscription price of the MONTHLY will be *three dollars net to all non-members of the* ASSOCIATION. The following adjustments for prospective members are proposed:

(1) Those who have already paid their subscriptions for the entire year 1916 are asked to send *one dollar* additional, which will entitle them to membership in the ASSOCIATION.

(2) Those who have not paid for 1916 are asked to send *three dollars*, which will entitle them to membership and include the MONTHLY. *No further subscriptions for 1916 will be received at the old rate of two dollars.*

In the case of subscriptions under (1) or (2) which expire *before* the end of 1916, please add *twenty cents extra* for each copy needed to complete the year. *Hereafter all subscriptions will date from January of each year.*

(3) An institution in which the Calculus is taught may become an *institutional member* of the ASSOCIATION by the payment of *five dollars* annually, which will entitle the library to receive *two copies of the* MONTHLY and the institution to send a voting delegate to all meetings of the ASSOCIATION. Institutions in which the Calculus is taught, whose libraries have already renewed their subscriptions for 1916, are asked to send *three dollars additional* and thus become institutional members of the ASSOCIATION.

Other institutions, and those not wishing to become institutional members, whose library subscriptions have already been renewed for 1916, are asked to send *one dollar additional* to complete the advanced price of the MONTHLY. No further subscriptions will be received at the old rate of two dollars, *and no discount from the advanced rate of three dollars will be allowed on subscriptions made through agencies.*

(4) The obligations of the MONTHLY for 1916 will, of course, be fulfilled on the former basis in the case of any individual or institution whose subscription has already been paid, and who may decline to make the adjustment on the new basis.

(5) Please note that all subscriptions to the MONTHLY and dues in the ASSOCIATION are to be paid to the SECRETARY-TREASURER, Professor W. D. CAIRNS, 55 East Lorain St., Oberlin, Ohio.

If you have not already returned the membership blank, please do so at once. Delay may make it impossible to secure the back issues of the MONTHLY.

CONSTITUTION AND BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

Article I—Name and Purpose.

1. This organization shall be known as The Mathematical Association of America.

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field.

Article II—Membership.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association; such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

Article III—Officers.

1. The officers of this Association shall be a President, two Vice-Presidents, a Secretary-Treasurer and twelve additional members of an Executive Council, together with a Committee of three on Publications, who shall be *ex-officio* members of the Council.

2. The President, Vice-Presidents and Secretary-Treasurer shall be elected annually for a term of one year, and four members of the Council shall be elected annually for a term of three years. They shall be eligible for reëlection, but not for more than two consecutive terms, except in the case of the Secretary-Treasurer, whose term may be extended indefinitely. The Committee on Publications, consisting of the Managing Editor and two other members, shall be appointed by the Council.

3. The Council shall transact the official business of the Association and shall report its actions at the annual meeting of the Association and in the official journal. Any proposed action of the Council which makes or alters a question of policy shall be published in the official journal before final action has been taken, so that members of the Association may make known to the Council their individual views.

4. The Council shall have authority to fill vacancies *ad interim*.

Article IV—Meetings.

1. The annual meeting of the Association shall be held at such time and place as the Council may direct.

2. The Council shall have power to call other meetings of the Association whenever it may be deemed expedient.

Article V—Sections.

1. Any group of members of this Association may petition the Council for authority to organize a Section of the Association for the purpose of holding local meetings. The Council shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such sections.

Article VI—Official Journal.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Council shall have power to conduct negotiations with respect to securing an official journal, and shall have full control of its publication and sale.

Article VII—Dues.

1. An individual member of the Association shall pay an initiation fee of two dollars at the time of his election.

The initiation fee shall be waived in case of those who join the Association before April 1, 1916, and this clause shall be dropped after its provisions have been fulfilled.

2. The annual dues of an individual member shall be three dollars, including a subscription to the official journal.

3. The annual dues of an institutional member shall be five dollars, including two subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1, of any year, shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

Article VIII—Amendments.

This Constitution may be amended at any annual meeting of the Association by a two-thirds vote of those present and voting, provided that such amendment shall have been printed in the official journal at least one month before the date of such meeting.

BY-LAWS.

1. *Election of Members.* Election to membership shall be by vote of the Council upon written application from the individual or institution seeking admission.

Those who shall be admitted to membership before April 1, 1916, shall constitute the list or charter members.

2. *Nomination and Election of Officers.* Two months before the date of the annual meeting, all members shall be given an opportunity to nominate by mail a candidate for each office for the ensuing year. One month before the annual meeting, the Council shall announce two candidates for each office, one being the person who received the highest vote in the nominations and the othef being selected by the Council from among the several nominees next in order.

The election shall be by mail or in person and shall close on the day of the annual meeting.

Twelve members of the Council shall be elected at the first meeting of the Association, and the secretary shall draw lots to determine which four of those elected shall serve for one, for two, and for three years respectively. (This clause shall be dropped after its provisions have been fulfilled.)

3. *Committees.* The Committee on Publications shall have charge of the official journal and of all other publications of the Association, under the direction of the Council.

The Council may appoint any other committees and delegate to them such power as may, in its judgment, seem desirable.

4. *Price of Publications.* The Council shall fix the price of the official journal, and of any other publications of the Association to non-members, but in no case shall the journal be sold for less than the annual dues of individual members, as specified in Article VII of the Constitution.

This shall not be construed to affect existing contracts with any subscribers or news agencies for the year 1916, who may decline to readjust on the new basis. (This clause shall be dropped after its provisions have been fulfilled.)

5. *Amendments.* These By-Laws may be amended at any annual meeting under the same conditions as specified in Article VIII of the Constitution.

IMPORTANT ANNOUNCEMENT

TO

ALL INTERESTED IN MATHEMATICAL PROGRESS

THE AMERICAN MATHEMATICAL MONTHLY, since its reorganization in January, 1913, has endeavored to fulfill its mission as "A JOURNAL FOR TEACHERS OF MATHEMATICS IN THE COLLEGIATE AND ADVANCED SECONDARY FIELDS."

A selection from the Tables of Contents thus far includes articles on—

The History of Mathematics, such as the following:

"History of the Exponential and Logarithmic Concepts," by PROFESSOR FLORIAN CAJORI of Colorado College;

"The Foundation Period in the History of Group Theory," by JOSEPHINE BURNS, Graduate Student at the University of Illinois;

"Errors in the Literature on Groups of Finite Order," by PROFESSOR G. A. MILLER, University of Illinois;

"Number Systems of the North American Indians," by PROFESSOR W. C. EELLS, United States Naval Academy;

"The Algebra of Abu Kamil," by PROFESSOR L. C. KARPINSKI, University of Michigan;

"Centers of Similitude of Circles and Certain Theorems Attributed to Monge. Were they known to the Greeks?" by PROFESSOR R. C. ARCHIBALD, Brown University;

"The History of Zeno's Arguments: Phases in the Development of the Theory of Limits," by PROFESSOR FLORIAN CAJORI, Colorado College.

Pedagogical Considerations, such as the following:

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